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SOME REMARKS ON ℓ^p AS AN n-NORMED SPACE

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Abstract. As in 2-normed spaces, we can also give two definitions for Cauchy sequences in n-normed spaces. It is known that in some cases, especially in the finite dimensional case and the standard case, two definitions are equivalent. What is not clear is in the infinite dimensional case. In this paper, we will prove that these two definitions are still equivalent in ℓ^p space.

1. Introduction

The concept of 2-normed spaces was first introduced by Gähler [1], while that of 2-inner product spaces was developed by Diminnie, Gähler and White [2]-[3]. Their generalization for $n \geq 2$ may be found in Misiak's works [4]-[5].

A 2-norm on a real vector space X of d dimension, where $d \geq 2$, is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following conditions for all $x,\ y,\ z \in X$ and for any $\alpha \in F$ (the field of X)

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$,
(3) $\|\alpha x, y\| = |\alpha|\, \|x, y\|$,
(4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space [1].

Let $(X, \langle .,. \rangle)$ be an inner product space, equipped with the standard 2-norm

$$\|x, y\|_S := \begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix}^{\frac{1}{2}}.$$

Note that geometrically $\|x, y\|$ represents the area of the parallelogram spanned by x and y. The determinant is known as the Gramian of x and y. Euclidean 2-norm on $\mathbb{R}^2$ is given by

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$$\|x,y\|_E = \text{abs}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right),\ x=(x_1,x_2),\ y=(y_1,y_2)\in\mathbb{R}^2,$$

where the subscript E is for Euclidean. The standard 2-norm is exactly same as the Euclidean 2-norm if $X=\mathbb{R}^2$ [6].

There are two definitions of Cauchy sequences in 2-normed spaces.

Definition 1.1. Let $\{y,z\}$ be a linearly independent set on a 2-normed space $(X,\|.,.\|)$. A sequence (x_k) in X is called a Cauchy with respect to the set $\{y,z\}$ if $\lim_{k,\,l\to\infty}\|x_k-x_l,y\|=0$ and $\lim_{k,\,l\to\infty}\|x_k-x_l,z\|=0$ [7].

Definition 1.2. A sequence (x_k) in a 2-normed space $(X,\|.,.\|)$ is called a Cauchy sequence with respect to the $\|.,.\|$ if $\lim_{k,\,l\to\infty}\|x_k-x_l,z\|=0$ for every nonzero $z\in X$ [8].

Definition 1.2 is clearly stronger than Definition 1.1.

Definition 1.3. A sequence (x_k) in a 2-normed space X is called a convergent sequence, if there is an x in X such that $\lim_{k\to\infty}\|x_k-x,z\|=0$ for every nonzero z in X [7].

In the light of the Definition 1.1, we can give another definition of convergent sequence in 2-normed space, clearly weaker than the Definition 1.3.

Definition 1.4. Let $\{y,z\}$ be a linearly independent set on a 2-normed space $(X,\|.,.\|)$. A sequence (x_k) in X is called convergent to some x in X with respect to the set $\{y,z\}$ if $\lim_{k\to\infty}\|x_k-x,y\|=0$ and $\lim_{k\to\infty}\|x_k-x,z\|=0$.

As in 2-normed spaces, we can also give two definitions for Cauchy sequences in n-normed spaces. It is known that in some cases, especially in the finite dimensional case and the standard case, two definitions are equivalent. What is not clear is in the infinite dimensional case. In this paper, we will prove that these two definitions are still equivalent in ℓ^p space.

2. Definitions and Preliminaries

Definition 2.1. Let $n\geq 2$ be an integer and X be a real vector space of dimension $d\geq n$ (d may be infinite). A real-valued function $\|.,...,.\|$ on X^n satisfying the following four properties for all $x_1,\ x_2,...,x_n,\ x,\ x'\in X$

(1) $\|x_1,x_2,...,x_n\|=0$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,
(2) $\|x_1,x_2,...,x_n\|$ is invariant under permutation,
(3) $\|\alpha x_1,x_2,...,x_n\|=|\alpha|\,\|x_1,x_2,...,x_n\|$, for any $\alpha\in\mathbb{R}$,
(4) $\|x+x',x_2,...,x_n\|\leq\|x,x_2,...,x_n\|+\|x',x_2,...,x_n\|$

is called an n-norm on X, and the pair $(X,\|.,...,.\|)$ is called an n-normed space [9].

For recent results on n-normed spaces, see, for example [4]-[18].

Example 2.1. Any real inner product space $(X, \langle ., . \rangle)$ can be equipped with the standard n-norm $\|x_1, ..., x_n\| := \sqrt{\det(\langle x_i, x_j \rangle)}$, which can be interpreted as the volume of the n-dimensional paralellepiped spanned by $x_1, ..., x_n \in X$. On $\mathbb{R}^n$, this n-norm can be simplified as $\|x_1, ..., x_n\| := |\det(x_i, x_j)|$ where $x_i = (x_{i1}, ..., x_{in}) \in \mathbb{R}^n$, $i = 1, ..., n$ [17].

Definition 2.2. A sequence (x_k) in an n-normed space $(X, \|., ..., .\|)$ is said to be convergent to some $x \in X$ in the n-norm if for each $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that $\|x_k - x, y_2, ..., y_n\| < \varepsilon$ for all $k \geq n_0$ and for every $y_2, ..., y_n \in X$ [9].

Definition 2.3. A sequence (x_k) in an n-normed space $(X, \|., ..., .\|)$ is said to be a Cauchy with respect to the n-norm if $\lim\limits_{k,l\to\infty} \|x_k - x_l, y_2, ..., y_n\| = 0$, for every $y_2, ..., y_n \in X$ [9].

As in the 2-normed spaces, we can give another definition of Cauchy sequences for n-normed spaces.

Definition 2.4. Let $A := \{a_1, ..., a_n\}$ be a linearly independent set on an n-normed space $(X, \|., ..., .\|)$. Then we say that a sequence (x_k) in X is said to be a Cauchy with respect to the set A if $\lim\limits_{k,l\to\infty} \|x_k - x_l, a_{i_2}, ..., a_{i_n}\| = 0$, for $\{i_2, ..., i_n\} \subset \{1, ..., n\}$.

Definition 2.3 is clearly stronger than Definition 2.4. But from the results in [10]-[11] we understand that the two definitions are still equivalent in ℓ^p and L^p (The space of all p-integrable functions) spaces. For other spaces, for example, for C[a,b] (the space of all continuous functions given on a closed interval $[a, b]$) we still don't know whether they are equivalent like ℓ^p and L^p. In the space $C[a, b]$ the equivalence is obtained for only some specific vectors, we don't know the equivalence for arbitrary vectors. This is an open problem also to explore. For details, we recommend readers to review the references [19]-[20]. E. Junaeti [19], in her thesis, used the tent functions (functions whose graphs look like a tent, that is; triangular shape) $a_1, ...a_n$ supported on the closed intervals $[a, a + h]$, $[a + h, a + 2h]$,...,$[a + (n-1)h, b]$, where $h = (b-a)/n$, and showed that if a sequence satisfies the Cauchy condition in the n-norm with respect to set $\{a_1, ..., a_n\}$, then it also satisfies the Cauchy condition in the usual supremum norm on $C[a, b]$.

We will show the equivalence of Definition 2.3 and Definition 2.4 for ℓ^p, which can be done similarly for L^p. Now we need some lemmas which were given in [10].

Lemma 2.1. *(Lemma 2.2, [10])* $\|x_1, ..., x_n\|_p \leq (n!)^{1-\frac{1}{p}} \|x_1\|_p ... \|x_n\|_p$ *holds for every* $x_1, ..., x_n \in \ell^p$.

Lemma 2.2. *(Proposition 2.3, [10]) Let* $\{a_1, ..., a_n\}$ *be a linearly independent set on* ℓ^p*. Then the following function*

$$\|x\|_p^* := \left[\sum_{\{i_2,...,i_n\}\subset\{1,...,n\}} \|x, a_{i_2}, ..., a_{i_n}\|_p^p \right]^{\frac{1}{p}}$$

defines a norm on ℓ^p.

Lemma 2.3. *(Proposition 2.5, [10]) Let $\{a_1, ..., a_n\}$ be a linearly independent set on ℓ^p. Then the norm $\|x\|_p^*$ is equivalent to the usual norm $\|x\|_p$ on ℓ^p. Precisely, for every $x \in \ell^p$ we have*

$$\frac{n\|a_1, ..., a_n\|_p}{(2n-1)\left[\|a_1\|_p + ... + \|a_n\|_p\right]}\|x\|_p \leq \|x\|_p^* \leq (n!)^{1-\frac{1}{p}}\left[\sum_{\{i_2,...,i_n\}\subset\{1,...,n\}} \|a_{i_2}\|_p^p ... \|a_{i_n}\|_p^p\right]^{\frac{1}{p}} \|x\|_p.$$

3. Main Results

Theorem 3.1. *The sequence (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.3 if and only if there exists a linearly independent set $A = \{a_1, ..., a_n\}$ such that the sequence (x_k) is a Cauchy sequence in ℓ^p with respect to the set A.*

Proof. Assume that (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.3. Then $\lim\limits_{k,l\to\infty} \|x_k - x_l, y_2, ..., y_n\|_p = 0$, for every $y_2, ..., y_n \in \ell^p$. Hence, there exists a linearly independent set $A = \{a_1, ..., a_n\}$ on ℓ^p, $\lim\limits_{k,l\to\infty} \|x_k - x_l, a_{i_2}, ..., a_{i_n}\|_p = 0$, for any $\{i_2, ..., i_n\} \subset \{1, ..., n\}$. Thus, we obtain the Definition 2.4.

Now, suppose that (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.4. Then for $\{i_2, ..., i_n\} \subset \{1, ..., n\}$ we have $\lim\limits_{k,l\to\infty} \|x_k - x_l, a_{i_2}, ..., a_{i_n}\|_p = 0$. Hence we obtain

$$\begin{aligned}\lim_{k,l\to\infty} \|x_k - x_l\|_p^* &= \left[\lim_{k,l\to\infty} \sum_{\{i_2,...,i_n\}\subset\{1,...,n\}} \|x_k - x_l, a_{i_2}, ..., a_{i_n}\|_p^p\right]^{\frac{1}{p}} \\ &= \left[\sum_{\{i_2,...,i_n\}\subset\{1,...,n\}} \lim_{k,l\to\infty} \|x_k - x_l, a_{i_2}, ..., a_{i_n}\|_p^p\right]^{\frac{1}{p}} \\ &= 0.\end{aligned}$$

By Lemma 2.3, we then conclude that $\lim\limits_{k,\, l\to\infty} \|x_k - x_l\|_p = 0$. Hence, for every $y_2, ..., y_n \in \ell^p$, we have by Lemma 2.1;

$$\|x_k - x_l, y_2, ..., y_n\|_p \leq (n!)^{1-\frac{1}{p}}\|x_k - x_l\|_p \|y2\|_p ... \|y_n\|_p.$$

Thus, we obtain $\lim\limits_{k,\, l\to\infty} \|x_k - x_l, y_2, ..., y_n\|_p = 0$ for every $y_2, ..., y_n \in \ell^p$. Hence, the Definition 2.3 is obtained. This completes the proof. □

Corollary 3.1. *Let $A := \{a_1, ..., a_n\}$ and $B := \{b_1, ..., b_n\}$ be linearly independent sets on ℓ^p. The sequence (x_k) is a Cauchy sequence with respect to the set A if and only if the sequence (x_k) is a Cauchy sequence with respect to the set B.*

Proof. Let (x_k) be a Cauchy sequence in ℓ^p with respect to the set A. Then from Theorem 3.1 (x_k) is a Cauchy sequence in ℓ^p according to Definition 2.3. Thus, we have from Theorem 3.1 that there exists a linearly independent set, i.e., say, $B = \{b_1, ..., b_n\}$ such that (x_k) is a Cauchy sequence in ℓ^p with respect to the set B.

For the converse, change the position of A and B. Hence, we have the result. □

Remark 3.1. As a result of the Corollary 3.1, Theorem 3.1 can be satisfied for an arbitrary linearly independent set on ℓ^p.

Remark 3.2. By replacing the phrases "(x_k) is Cauchy" with "(x_k) converges to x" and "$x_k - x_l$" with "$x_k - x$ ", we see that the analogues of Definition 2.4, Definition 2.3, Theorem 3.1 and Corollary 3.1 hold for convergent sequences.

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