ON SOME RESULTS OF $I_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

ERDİNÇ DÜNDAR

(Communicated by Nihal YILMAZ ÖZGÜR)

Abstract. In this work, we investigate some results of $I_2$-convergence of double sequences of real valued functions and prove a decomposition theorem.

1. BACKGROUND AND INTRODUCTION

The concept of convergence of a real sequence was independently extended to statistical convergence by Fast [11] and Schoenberg [30]. This concept was extended to the double sequences by Mursaleen and Edely [21]. A lot of developments have been made in this area after the works of Šalát [29] and Fridy [13, 14]. Furthermore Gökhan et al. [16] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [11, 13, 14, 28]. Çakan and Altay [5] presented multidimensional analogues of the results presented by Fridy and Orhan [12].

The idea of $I$-convergence was introduced by Kostyrko et al. [18] as a generalization of statistical convergence which is based on the structure of the ideal $I$ of subset of the set of natural numbers. Nuray and Ruckle [26] independently introduced the same concept with another name generalized statistical convergence. Kostyrko et al. [19] gave some of basic properties of $I$-convergence and dealt with extremal $I$-limit points. Das et al. [6] introduced the concept of $I$-convergence of double sequences in a metric space and studied some of its properties. Also Das and Malik [7] introduced the concept of $I$-limit points, $I$-cluster points and $I$-limit superior and $I$-limit inferior of double sequences. Balcerzak et al. [4] discussed various kinds of statistical convergence and $I$-convergence of sequences of functions with values in $\mathbb{R}$ or in a metric space. Gezer and Karakus [15] investigated $I$-pointwise and uniform convergence and $I^*$-pointwise and uniform convergence of function sequences and then they examined the relation between them. Dündar and Altay [8] studied the
ON SOME RESULTS OF $I_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

concepts of $I_2$-Cauchy and $I_2^*$-Cauchy for double sequences in a linear metric space and investigated the relation between $I_2$-convergence and $I_2^*$-convergence of double sequences of functions defined between linear metric spaces. Also, some results on $I$-convergence may be found in [2, 3, 9, 20, 22, 23, 24, 25, 31].

In this study, we investigate some results of $I_2$-convergence of double sequences of real valued functions and prove a decomposition theorem for $I_2$-convergent double sequences.

2. DEFINITIONS AND NOTATIONS

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively.

Now, we recall the concept of statistical and ideal convergence of the sequences (See [6, 8, 10, 11, 16, 18, 21, 27]).

A double sequence $x = (x_{mn})_{m,n \in \mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n \to \infty} x_{mn} = L.$$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let $K_{mn}$ be the number of $(j,k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\{K_{mn}\}$ has a limit in Pringsheim’s sense then we say that $K$ has double natural density and is denoted by

$$d_2(K) = \lim_{m,n \to \infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})_{m,n \in \mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to $f$ on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N(x,\varepsilon)$ such that

$$|f_{mn}(x) - f(x)| < \varepsilon,$$

for all $m,n > N$. In this case we write

$$\lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \to f, \text{ as } m,n \to \infty,$$

for each $x \in S$.

A double sequence of functions $\{f_{ij}\}$ is said to be pointwise statistically convergent to $f$ on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{m,n \to \infty} \frac{1}{mn} |\{(i,j), i \leq m \text{ and } j \leq n : |f_{ij}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$,

$$|f_{ij}(x) - f(x)| < \varepsilon, \text{ a.a.}(i,j).$$

In this case we write

$$st - \lim_{i,j \to \infty} f_{ij}(x) = f(x) \text{ or } f_{ij} \overset{st}{\to} f,$$

for each $x \in S$.

Let $X \neq \emptyset$. A class $I$ of subsets of $X$ is said to be an ideal in $X$ provided:
Lemma 2.1. [18] If $\mathcal{I}$ is a nontrivial ideal in $X$, $X \neq \emptyset$, then the class
\[ \mathcal{F}(\mathcal{I}) = \{ M \subset X : (\exists A \in \mathcal{I})(M = X\setminus A) \} \]
is a filter on $X$, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in $X$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is also admissible.

Let $\mathcal{I}_2^0 = \{ A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A) \}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let $(X, \rho)$ be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal.

A double sequence $x = (x_{mn})_{m,n \in \mathbb{N}}$ of elements of $X$ is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have
\[ A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2. \]

In this case we say that $x$ is $\mathcal{I}_2$-convergent and we write
\[ \mathcal{I}_2 - \lim_{m,n \to \infty} x_{mn} = L. \]

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let $(X, \rho)$ be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal.

A double sequence $x = (x_{mn})$ of elements of $X$ is said to be $\mathcal{I}_2^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$) such that
\[ \lim_{m,n \to \infty} x_{mn} = L, \]

for $(m, n) \in M$ and we write
\[ \mathcal{I}_2^* - \lim_{m,n \to \infty} x_{mn} = L. \]

Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-convergent to $f$ on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$
\[ \{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}_2, \]

for each (fixed) $x \in S$. This can be written by the formula
\[ (\forall x \in S) \ (\forall \varepsilon > 0) \ (\exists H \in \mathcal{I}_2) \ (\forall (m, n) \notin H) \ |f_{mn}(x) - f(x)| < \varepsilon. \]

This is written as
\[ f_{mn} \mathcal{I}_2 \to f, \text{ as } m, n \to \infty. \]

The function $f$ is called the double $\mathcal{I}_2$-limit (or Pringsheim $\mathcal{I}_2$-limit) function of the $\{f_{mn}\}$. 
ON SOME RESULTS OF $\mathcal{I}_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise $\mathcal{I}_2^*$-convergent to $f$ on $S \subset \mathbb{R}$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e. $\mathbb{N} \times \mathbb{N}\setminus M \in \mathcal{I}_2$) such that

$$\lim_{m,n \to \infty} f_{mn}(x) = f(x),$$

for $(m,n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \xrightarrow{\mathcal{I}_2^*} f, \text{ as } m,n \to \infty,$$

for each $x \in S$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the property $(AP2)$, if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j \in \mathcal{I}_2$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Now we begin with quoting the lemmas due to Dndar and Altay [8, 10] which are needed throughout the paper.

**Lemma 2.2** ([10]). Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ is a double sequence of functions and $f$ be a function on $S \subset \mathbb{R}$. Then

$$\mathcal{I}_2^* - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ implies } \mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x),$$

for each $x \in S$.

**Lemma 2.3** ([8]). Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal having the property $(AP2)$, $(X,d_x)$ and $(Y,d_y)$ two linear metric spaces, $f_{mn} : X \to Y$ a double sequence of functions and $f : X \to Y$. If $\{f_{mn}\}$ double sequence of functions is $\mathcal{I}_2$-convergent, then it is $\mathcal{I}_2^*$-convergent.

3. SOME RESULTS OF $\mathcal{I}_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

Throughout the paper we use convergence instead of pointwise convergence.

**Theorem 3.1.** Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ be a double sequence of functions and $f$ be a function on $S \subset \mathbb{R}$. If $c \in \mathbb{R}$ and $\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x)$, then we have

$$\mathcal{I}_2 - \lim_{m,n \to \infty} cf_{mn}(x) = cf(x),$$

for each $x \in S$.

**Proof.** Let $c \in \mathbb{R}$ and $\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x)$, for each $x \in S$. If $c = 0$, there is nothing to prove, so we assume that $c \neq 0$.

Let $\varepsilon > 0$ be given. Then,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : |cf_{mn}(x) - cf(x)| \geq \varepsilon\right\} \subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{|c|}\right\} \in \mathcal{I}_2.$$

Hence, $\mathcal{I}_2 - \lim_{m,n \to \infty} cf_{mn}(x) = cf(x)$ for each $x \in S$. \qed
Theorem 3.2. Let \( \mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal, \( \{f_{mn}\} \) and \( \{g_{mn}\} \) be two double sequences of functions, \( f \) and \( g \) be two functions on \( S \subset \mathbb{R} \) and

\[
\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ and } \mathcal{I}_2 - \lim_{m,n \to \infty} g_{mn}(x) = g(x),
\]

for each \( x \in S \). Then, we have

(i) \( \mathcal{I}_2 - \lim_{m,n \to \infty} (f_{mn} + g_{mn})(x) = f(x) + g(x) \),

(ii) \( \mathcal{I}_2 - \lim_{m,n \to \infty} (f_{mn}g_{mn})(x) = f(x)g(x) \),

for each \( x \in S \).

Proof. (i) Let \( \varepsilon > 0 \) be given. Since

\[
\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ and } \mathcal{I}_2 - \lim_{m,n \to \infty} g_{mn}(x) = g(x),
\]

therefore

\[
A\left(\frac{\varepsilon}{2}\right) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2}\} \in \mathcal{I}_2
\]

and

\[
B\left(\frac{\varepsilon}{2}\right) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |g_{mn}(x) - g(x)| \geq \frac{\varepsilon}{2}\} \in \mathcal{I}_2,
\]

for each \( x \in S \) and by definition of ideal we have \( A\left(\frac{\varepsilon}{2}\right) \cup B\left(\frac{\varepsilon}{2}\right) \in \mathcal{I}_2 \). Now define the set

\[
C(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x))| \geq \varepsilon\}
\]

and it is sufficient to prove that \( C(\varepsilon) \subset A\left(\frac{\varepsilon}{2}\right) \cup B\left(\frac{\varepsilon}{2}\right) \), for each \( x \in S \). Let \( (m, n) \in C(\varepsilon) \), then we have

\[
\varepsilon \leq |(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x))| \\
\leq |f_{mn}(x) - f(x)| + |g_{mn}(x) - g(x)|,
\]

for each \( x \in S \). As both of \( |f_{mn}(x) - f(x)|, |g_{mn}(x) - g(x)| \) can not be (together) strictly less than \( \frac{\varepsilon}{2} \), and therefore we have either

\[
|f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2} \text{ or } |g_{mn}(x) - g(x)| \geq \frac{\varepsilon}{2},
\]

for each \( x \in S \). This shows that

\[
(m,n) \in A\left(\frac{\varepsilon}{2}\right) \text{ or } (m,n) \in B\left(\frac{\varepsilon}{2}\right)
\]

and so we have

\[
(m,n) \in A\left(\frac{\varepsilon}{2}\right) \cup B\left(\frac{\varepsilon}{2}\right).
\]

Hence, \( C(\varepsilon) \subset A\left(\frac{\varepsilon}{2}\right) \cup B\left(\frac{\varepsilon}{2}\right) \).

(ii) Since \( \mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \), therefore for \( \varepsilon = 1 > 0 \)

\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq 1\} \in \mathcal{I}_2
\]

for each \( x \in S \) and so

\[
A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| < 1\} \in \mathcal{F}(\mathcal{I}_2)
\]

for each \( x \in S \). Also for any \( (m, n) \in A \)

\[
|f_{mn}(x)| < 1 + f(x),
\]

for each \( x \in S \). Let \( \varepsilon > 0 \) be given. Choose \( \delta > 0 \) such that

\[
0 < 2\delta < \frac{\varepsilon}{|f| + |g| + 1}.
\]
It follows from the assumption that
\[ B = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| < \delta \} \in \mathcal{F}(I_2) \]
and
\[ C = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : |g_{mn}(x) - g(x)| < \delta \} \in \mathcal{F}(I_2), \]
for each \( x \in S \). Since \( \mathcal{F}(I_2) \) is a filter, therefore \( A \cap B \cap C \in \mathcal{F}(I_2) \). Then for each \((m, n) \in A \cap B \cap C\) we have
\[ |f_{mn}(x)g_{mn}(x) - f(x)g(x)| = |f_{mn}(x)g_{mn}(x) - f_{mn}(x)g(x) + f_{mn}(x)g(x) - f(x)g(x)| \]
\[ \leq |f_{mn}(x)||g_{mn}(x) - g(x)| + |g(x)||f_{mn}(x) - f(x)| \]
\[ < (|f(x)| + 1)\delta + |g(x)|\delta = (|f(x)| + |g(x)| + 1)\delta < \varepsilon, \]
for each \( x \in S \). Hence we have
\[ \{ (m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x)g_{mn}(x) - f(x)g(x)| \geq \varepsilon \} \in I_2, \]
for each \( x \in S \). This completes the proof of theorem. \( \square \)

Now, we give the decomposition theorem for double sequences of functions.

**Theorem 3.3.** Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal having the property (AP2), \( \{ f_{mn} \} \) be a double sequence of functions and \( f \) be a function on \( S \subset \mathbb{R} \). Then the following conditions are equivalent:

(i) \( I_2 = \lim_{m,n \to \infty} f_{mn}(x) = f(x) \), for each \( x \in S \).

(ii) There exist \( \{ g_{mn} \} \) and \( \{ h_{mn} \} \) be two double sequences of functions such that
\[ f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \quad \lim_{m,n \to \infty} g_{mn}(x) = f(x) \quad \text{and} \quad \supp h_{mn}(x) \in I_2, \]
for each \( x \in S \), where \( \supp h_{mn}(x) = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0 \} \).

**Proof.** (i) \( \Rightarrow \) (ii): \( I_2 = \lim_{m,n \to \infty} f_{mn}(x) = f(x) \) for each \( x \in S \). Then by Lemma 2.3 there exists a set \( M \in \mathcal{F}(I_2) \) (i.e., \( H = \mathbb{N} \times \mathbb{N} \backslash M \in I_2 \)) such that
\[ \lim_{(m,n)\in M} f_{mn}(x) = f(x), \]
for each \( x \in S \). Let us define the double sequence \( \{ g_{mn} \} \) by
\[ g_{mn}(x) = \begin{cases} f_{mn}(x), & (m,n) \in M \\ f(x), & (m,n) \in \mathbb{N} \times \mathbb{N} \backslash M. \end{cases} \] (3.1)
It is clear that \( \{ g_{mn} \} \) is a double sequence of functions on \( S \) and
\[ \lim_{m,n \to \infty} g_{mn}(x) = f(x), \]
for each \( x \in S \). Also let
\[ h_{mn}(x) = f_{mn}(x) - g_{mn}(x), \quad m,n \in \mathbb{N}, \]
for each \( x \in S \). Since
\[ \{ (m, n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \neq g_{mn}(x) \} \subset \mathbb{N} \times \mathbb{N} \backslash M \in I_2, \]
for each \( x \in S \), so we have
\[ \{ (m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0 \} \in I_2. \]
It follows that \( \supp h_{mn}(x) \in I_2 \) and by (3.1) and (3.2) we get \( f_{mn}(x) = g_{mn}(x) + h_{mn}(x) \), for each \( x \in S \).
(ii) $\Rightarrow$ (i): Suppose that there exist two sequences $\{g_{mn}\}$ and $\{h_{mn}\}$ on $S$ such that
\begin{equation}
 f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \quad \lim_{m,n \to \infty} g_{mn}(x) = f(x), \quad \text{supp } h_{mn}(x) \in \mathcal{I}_2,
\end{equation}
for each $x \in S$, where $\text{supp } h_{mn}(x) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\}$. We will show that
\[ \mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x), \]
for each $x \in S$. Let
\[ M = \{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) = 0\} = \mathbb{N} \times \mathbb{N} \setminus \text{supp } h_{mn}(x). \]
Since
\[ \text{supp } h_{mn}(x) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2, \]
from (3.3) and (3.4) we have $M \in F(\mathcal{I}_2)$, $f_{mn}(x) = g_{mn}(x)$ for $(m, n) \in M$ and
\[ \mathcal{I}_2 - \lim_{(m,n) \in M} f_{mn}(x) = f(x), \]
for each $x \in S$. By Lemma 2.2 it follows that
\[ \mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x), \]
for each $x \in S$. This completes the proof. \hfill \Box

**Corollary 3.1.** Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal having the property $(AP2)$, $\{f_{mn}\}$ be a double sequence of functions and $f$ be a function on $S \subset \mathbb{R}$. Then
\[ \mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \]
if and only if there exist two double sequences $\{g_{mn}\}$ and $\{h_{mn}\}$ of functions on $S$ such that
\begin{equation}
 f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \quad \lim_{m,n \to \infty} g_{mn}(x) = f(x), \quad \text{and } \mathcal{I}_2 - \lim_{m,n \to \infty} h_{mn}(x) = 0,
\end{equation}
for each $x \in S$.

**Proof.** Let $\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x)$ and $\{g_{mn}\}$ is the sequence defined by (3.1). Consider the sequence
\begin{equation}
 h_{mn}(x) = f_{mn}(x) - g_{mn}(x), \quad m, n \in \mathbb{N},
\end{equation}
for each $x \in S$. Then we have
\[ \lim_{m,n \to \infty} g_{mn}(x) = f(x) \]
and since $\mathcal{I}_2$ is a strongly admissible ideal so
\[ \mathcal{I}_2 - \lim_{m,n \to \infty} g_{mn}(x) = f(x), \]
for each $x \in S$. By Theorem 3.2 and by (3.5) we have
\[ \mathcal{I}_2 - \lim_{m,n \to \infty} h_{mn}(x) = 0, \]
for each $x \in S$.

Now let $f_{mn}(x) = g_{mn}(x) + h_{mn}(x)$, where
\[ \lim_{m,n \to \infty} g_{mn}(x) = f(x) \quad \text{and } \mathcal{I}_2 - \lim_{m,n \to \infty} h_{mn}(x) = 0,$
for each $x \in S$. Since $\mathcal{I}_2$ is a strongly admissible ideal so
\[
\mathcal{I}_2 - \lim_{m,n \to \infty} g_{mn}(x) = f(x)
\]
and by Theorem 3.2 we get
\[
\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x),
\]
for each $x \in S$.

**Remark 3.1.** In Theorem 3.3, if (ii) is satisfied then the strongly admissible ideal $\mathcal{I}_2$ need not have the property $(AP_2)$. Since
\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : |h_{mn}(x)| \geq \varepsilon\} \subset \{(m,n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2
\]
for each $\varepsilon > 0$, then
\[
\mathcal{I}_2 - \lim_{m,n \to \infty} h_{mn}(x) = 0.
\]
Thus, we have
\[
\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x),
\]
for each $x \in S$.

Acknowledgements

The author would like to express his thanks to Professor Bilâl Altay, Faculty of Education, İnönü University, 44280-Malatya, TURKEY for his careful reading of an earlier version of this paper and the constructive comments which improved the presentation of the paper.

References


[9] E. Dündar, B. Altay, $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences, (under communication).


