A NOTE ON LAGUERRE MATRIX POLYNOMIALS

ALİ ÇEVİK AND ABDULLAH ALTIN

(Communicated by İ. Onur KIYMAZ)

Abstract. In this paper, some new relations for Laguerre matrix polynomials are given.

1. Introduction

Recently, matrix polynomials that are solutions of a second order matrix differential equation are very popular subject in mathematics. In this area, many papers have been published ([17],[16],[18],[11],[19],[10],[23]). Many properties, extensions and generalizations of them have been introduced ([20],[9],[13],[12],[22],[15],[4],[24],[2],[3],[1],[8]). Laguerre matrix polynomial is one of them ([21],[20],[7],[5],[6]).

In this paper, firstly a few lemmas are given. After, some new relations for Laguerre matrix polynomials are obtained by using these lemmas.

Throughout this paper, for a matrix $A \in \mathbb{C}^{r \times r}$, $\sigma(A)$ denotes the set of all eigenvalues of $A$ and is called its spectrum. $A$ is a positive stable matrix if $\text{Re}(\lambda) > 0$ for all $\lambda \in \sigma(A)$.

If $A_0, A_1, ..., A_n$ are elements of $\mathbb{C}^{r \times r}$ and $A_n \neq 0$, then

$$ P(x) = A_n x^n + A_{n-1} x^{n-1} + ... + A_1 x + A_0 $$

is called a matrix polynomial of degree $n$ in $x$ for every integer $n \geq 0$. From [17],

$$ (A)_n = A(A + I)(A + 2I)...(A + (n-1)I) ; \quad n \geq 1 ; \quad (A)_0 = I. $$

is written. Using (1.1), we see that

$$ \frac{I}{(n-k)!} = (-1)^k \frac{(-nI)_k}{n!} ; \quad 0 \leq k \leq n. $$

In [17], if $f(z)$ and $g(z)$ are holomorphic functions which are defined in an open set $\Omega$ of the complex plane, and if $A$ is a matrix in $\mathbb{C}^{r \times r}$ for which $\sigma(A) \subset \Omega$, using the properties of the matrix functional calculus in [14] then

$$ f(A)g(A) = g(A)f(A). $$

Date: Received: April 17, 2015; Revised: May 1, 2015; Accepted: May 13, 2015.

2010 Mathematics Subject Classification. 33C45 (15A60, 15A09).

Key words and phrases. Laguerre matrix polynomials.

The authors are thankful for the referees for their helpful comments.
Hence, if \( B \in \mathbb{C}^{r \times r} \) is a matrix for which \( \sigma(B) \subset \Omega \) and \( AB = BA \), then
\[
f(A)g(B) = g(B)f(A).
\]

Let \( A \) be a matrix in \( \mathbb{C}^{r \times r} \) satisfying \((-k) \notin \sigma(A)\) for every integer \( k > 0 \) and \( \lambda \) be a complex number whose real part is positive. In [17], \( n \)-th degree Laguerre matrix polynomial, \( L_n^{(A,\lambda)}(x) \) is defined by
\[
L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} (A + I)_n (A + I)^{-1}_k (\lambda x)^k.
\]

By using (1.2), \( L_n^{(A,\lambda)}(x) \) can be written in the form
\[
L_n^{(A,\lambda)}(x) = \frac{(A + I)_n}{n!} \sum_{k=0}^{n} (-nI)_k (A + I)_k^{-1} \frac{(\lambda x)^k}{k!}.
\]

Also, Laguerre matrix polynomials have the following derivative relation [20],
\[
\frac{d}{dx} L_n^{(A,\lambda)}(x) = -\lambda L_{n-1}^{(A+I,\lambda)}(x), \quad n \geq 1.
\]

Lemma 1.1. [7] The raising operator for Laguerre matrix polynomials is
\[
\frac{d}{dx} x A e^{-\lambda x} L_n^{(A,\lambda)}(x) = (n + 1) x A^{-1} e^{-\lambda x} L_{n+1}^{(A-I,\lambda)}(x), \quad x > 0
\]
where \( A \) is positive stable matrix in \( \mathbb{C}^{r \times r} \) and \( \text{Re} (\lambda) > 0 \).

2. Some new relations for Laguerre matrix polynomials

Lemma 2.1. Let \( A \) be a matrix in \( \mathbb{C}^{r \times r} \) satisfying the spectral condition
\[
\text{Re} (\mu) > 1 \quad \text{for all} \quad \mu \in \sigma(A),
\]
and \( \lambda \) be a complex number with \( \text{Re} (\lambda) > 0 \). For Laguerre matrix polynomials,
\[
\frac{d}{dx} x^A L_n^{(A,\lambda)}(x) = (A + I)n x^{A-I} L_n^{(A-I,\lambda)}(x), \quad x > 0
\]
is satisfied.

Proof. We start by taking the derivative of \( x^A L_n^{(A,\lambda)}(x) \) with respect to \( x \). Thus, we have
\[
\frac{d}{dx} x^A L_n^{(A,\lambda)}(x) = \frac{d}{dx} \left[ x^A (A + I)_n \sum_{k=0}^{n} (-nI)_k (A + I)_k^{-1} \left( \frac{\lambda x}{k!} \right)^k \right]
\]
\[
= \frac{d}{dx} \left[ \frac{1}{n!} \sum_{k=0}^{n} \lambda^k (-nI)_k (A + I)_n (A + I)_k^{-1} \left( x^{A+k} \right) \right]
\]
\[
= \frac{1}{n!} \sum_{k=0}^{n} \left\{ \lambda^k (-nI)_k (A + I)_n (A + I)_k^{-1} (A + kI) \frac{x^{A+(k-1)}I}{k!} \right\}.
\]
Then by using (1.1),
\[
\frac{d}{dx} \left[ x^A L_n^{(A,\lambda)}(x) \right] = \frac{x^{A-I}}{n!} \sum_{k=0}^{n} \lambda^k (-nI)_k (A)_n (A+nI)_k^{-1} \frac{x^k}{k!}
\]
\[
= (A+nI) x^{A-I} \frac{d}{dx} \left[ L_n^{(A,\lambda)}(x) \right] \frac{(\lambda x)^k}{k!}
\]
holds. This completes the proof. \(\square\)

**Theorem 2.1.** Let \(A\) be a matrix in \(\mathbb{C}^{r \times r}\) satisfying the spectral condition (2.1) and \(\text{Re}(\lambda) > 0\). Laguerre matrix polynomials satisfy the following relation with \(x > 0\)
\[
AL_n^{(A,\lambda)}(x) = (A+nI)L_n^{(A-I,\lambda)}(x) + \lambda x L_{n-1}^{(A+I,\lambda)}(x) , \ n \geq 1.
\]

**Proof.** The derivative of multiplication of \(x^A L_n^{(A,\lambda)}(x)\) with respect to \(x\) is, as follows from (1.3),
\[
\frac{d}{dx} \left[ x^A L_n^{(A,\lambda)}(x) \right] = Ax^{A-I} L_n^{(A,\lambda)}(x) + x^A \frac{d}{dx} L_n^{(A,\lambda)}(x)
\]
\[
= Ax^{A-I} L_n^{(A,\lambda)}(x) - \lambda x L_{n-1}^{(A+I,\lambda)}(x) , \ n \geq 1.
\]
Using (2.2) in the left side of this equation,
\[
Ax^{A-I} L_n^{(A,\lambda)}(x) = (A+nI) x^{A-I} L_n^{(A-I,\lambda)}(x) + \lambda x L_{n-1}^{(A+I,\lambda)}(x) , \ n \geq 1
\]
is written. Then multiplying both sides with the inverse of \(x^{A-I}\),
\[
AL_n^{(A,\lambda)}(x) = (A+nI)L_n^{(A-I,\lambda)}(x) + \lambda x L_{n-1}^{(A+I,\lambda)}(x) , \ n \geq 1
\]
is obtained. \(\square\)

**Theorem 2.2.** Let \(A\) be a matrix in \(\mathbb{C}^{r \times r}\) satisfying the spectral condition (2.1) and \(\text{Re}(\lambda) > 0\). For Laguerre matrix polynomials,
\[
(n+1) L_{n+1}^{(A-I,\lambda)}(x) = (A+nI) L_n^{(A-I,\lambda)}(x) - \lambda x L_n^{(A,\lambda)}(x) , \ x > 0
\]
holds.

**Proof.** Starting from the derivative of \(e^{-\lambda x} x^A L_n^{(A,\lambda)}(x)\) with respect to \(x\) and using Lemma 2.1, we can write
\[
\frac{d}{dx} \left[ e^{-\lambda x} x^A L_n^{(A,\lambda)}(x) \right] = -\lambda e^{-\lambda x} x^A L_n^{(A,\lambda)}(x) + e^{-\lambda x} \frac{d}{dx} \left[ x^A L_n^{(A,\lambda)}(x) \right]
\]
\[
= -\lambda e^{-\lambda x} x^A L_n^{(A,\lambda)}(x) + (A+nI) e^{-\lambda x} x^{A-I} L_n^{(A-I,\lambda)}(x).
\]
Combining (1.4) and (2.3), then multiplying both side with \(e^{\lambda x} x^{-A+I}\), the proof is completed. \(\square\)
REFERENCES


(A. ÇEVİK) MERSIN UNIVERSITY, DEPARTMENT OF MATHEMATICS, ÇİFTLİKKÖY TR-33343 MERSİN, TURKEY
E-mail address: cevik@mersin.edu.tr

(A. ALTIN) ANKARA UNIVERSITY, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, TANDOĞAN TR-06100 ANKARA, TURKEY
E-mail address: altin@science.ankara.edu.tr