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On new inequalities of Hermite-Hadamard-Fejer type for quasi-geometrically convex functions via fractional integrals

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Abstract
In this paper, new Hermite-Hadamard-Fejer type integral inequalities for quasi-geometrically convex functions in fractional integral forms are obtained.

Keywords: *Hermite-Hadamard inequality; Hermite-Hadamard-Fejer inequality; Hadamard fractional integral; quasi-geometrically convex function.*

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [2].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or their weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a + b)/2$.

For some results which generalize, improve and extend the inequalities (1.1) and (1.2) see [3, 12–14].

Definition 1. *[10, 11]. A function $f : I \subset (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if*

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

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Definition 2. *[5]. A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be quasi-geometrically convex on I if*

$$f(x^t y^{1-t}) \leq \sup\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$.

In [9], Latif et al. established following Hermite-Hadamard-Fejer type inequality for GA-convex functions as follows:

Theorem 2. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. Let $g:[a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. Then*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx \leq \int_a^b \frac{f(x)g(x)}{x}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x}dx. \tag{1.3}$$

In [8], Kunt et al. established following Hermite-Hadamard and Hermite-Hadamard-Fejer type inequality for GA-convex function in fractional integral forms as follows:

Theorem 3. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function with $a < b$ and $f \in L[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.4}$$

with $\alpha > 0$.

Theorem 4. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals holds:*

$$\begin{aligned} f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] &\leq \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] \end{aligned} \tag{1.5}$$

with $\alpha > 0$.

Lemma 1. *[8]. Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$ then the following equality for fractional integrals holds:*

$$\left[\begin{array}{c} f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] \\ -\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right] \end{array}\right] = \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln\frac{s}{a}\right)^{\alpha-1} g(s)\frac{ds}{s}\right) f'(t)\,dt \\ +\int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln\frac{b}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}\right) f'(t)\,dt \end{array}\right] \tag{1.6}$$

with $\alpha > 0$.

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 3. *[7]. Let $f \in L[a,b]$. The Hadamard integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x > a \text{ and } J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [4–6, 15, 16].

In this paper, we obtain some Hermite-Hadamard-Fejer type integral inequalities for quasi-geometrically convex functions in fractional integral forms.

2. Main results

Throughout this section, we write $\|g\|_{\infty}=\sup_{t\in[a,b]}|g(t)|$, for the continuous function $g:[a,b]\to\mathbb{R}$.

Theorem 5. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, the interior of I, such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|$ is quasi-geometrically convex on $[a,b]$, $g:[a,b]\to\mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality for fractional integrals holds:*

$$\left|\begin{array}{l} f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right] \\ -\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\end{array}\right| \leq \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}C_1(\alpha)\sup\left\{|f'(a)|,|f'(b)|\right\}$$

where

$$C_1(\alpha)=\int_0^{\frac{1}{2}}u^{\alpha}\left[a^{1-u}b^{u}+a^{u}b^{1-u}\right]du,$$

with $\alpha>0$.

Proof. From Lemma 1 we have

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left[\begin{array}{l}\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|\,dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|\,dt\end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{l}\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|\,dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|\,dt\end{array}\right].
\end{aligned}$$

Setting $t=a^{1-u}b^{u}$ and $dt=a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du$ gives

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{l}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^{u}}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du\end{array}\right] \\
=\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{l}\int_0^{\frac{1}{2}}\left(\left.\frac{\left(\ln\frac{s}{a}\right)^{\alpha}}{\alpha}\right|_a^{a^{1-u}b^{u}}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\left.\frac{-\left(\ln\frac{b}{s}\right)^{\alpha}}{\alpha}\right|_{a^{1-u}b^{u}}^b\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du\end{array}\right] \\
=\ & \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{l}\int_0^{\frac{1}{2}}u^{\alpha}\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)du \\ +\int_{\frac{1}{2}}^1(1-u)^{\alpha}\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)du\end{array}\right]. \qquad (2.1)
\end{aligned}$$

Since $|f'|$ is quasi-geometrically convex on $[a,b]$, we have

$$\left|f'(a^{1-u}b^{u})\right|\leq\sup\left\{|f'(a)|,|f'(b)|\right\}. \qquad (2.2)$$

If we use (2.2) in (2.1), we have

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\
\leq\ & \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{l}\int_0^{\frac{1}{2}}u^{\alpha}\sup\left\{|f'(a)|,|f'(b)|\right\}\left(a^{1-u}b^{u}\right)du \\ +\int_{\frac{1}{2}}^1(1-u)^{\alpha}\sup\left\{|f'(a)|,|f'(b)|\right\}\left(a^{1-u}b^{u}\right)du\end{array}\right] \\
=\ & \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{l}\int_0^{\frac{1}{2}}u^{\alpha}\left(a^{1-u}b^{u}\right)du \\ +\int_0^{\frac{1}{2}}u^{\alpha}\left(a^{u}b^{1-u}\right)du\end{array}\right]\sup\left\{|f'(a)|,|f'(b)|\right\} \\
=\ & \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\int_0^{\frac{1}{2}}u^{\alpha}\left[a^{1-u}b^{u}+a^{u}b^{1-u}\right]du\right]\sup\left\{|f'(a)|,|f'(b)|\right\}.
\end{aligned}$$

This completes the proof. □

Corollary 1. *In Theorem 5:*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer inequality for quasi-geometrically convex functions which is related to the left-hand side of (1.3):

$$\left| f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x)\, g(x)}{x} dx \right| \leq \|g\|_\infty \ln^2\left(\frac{b}{a}\right) C_1(1) \sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\},$$

(2) If we take $g(x) = 1$ we have the following Hermite-Hadamard inequality for quasi-geometrically convex functions in fractional integral forms which is related to the left-hand side of (1.4):

$$\left| f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^\alpha}\left[J^\alpha_{\sqrt{ab}-} f(a) + J^\alpha_{\sqrt{ab}+} f(b)\right] \right| \leq \frac{\ln\left(\frac{b}{a}\right)}{2^{1-\alpha}} C_1(\alpha) \sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\},$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard inequality for quasi-geometrically convex functions:

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right| \leq \ln\left(\frac{b}{a}\right) C_1(1) \sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\}.$$

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $f' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|f'|^q$, $q \geq 1$, is quasi-geometrically convex on $[a,b]$, $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality for fractional integrals holds:*

$$\left[\left| \begin{array}{c} f\left(\sqrt{ab}\right)\left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b)\right] \\ -\left[J^\alpha_{\sqrt{ab}-}(fg)(a) + J^\alpha_{\sqrt{ab}+}(fg)(b)\right] \end{array} \right|\right] \leq \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)(\alpha+1)^{\frac{1}{q}-1}}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\Gamma(\alpha+1)} C_2(\alpha)\left[\sup\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}$$

where

$$C_2(\alpha) = \left[\int_0^{\frac{1}{2}} u^\alpha \left(a^{1-u} b^u\right)^q du\right]^{\frac{1}{q}} + \left[\int_{\frac{1}{2}}^1 (1-u)^\alpha \left(a^{1-u} b^u\right)^q du\right]^{\frac{1}{q}},$$

with $\alpha > 0$.

Proof. Using Lemma 1, we have

$$\begin{aligned} &\left| f\left(\sqrt{ab}\right)\left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b)\right] - \left[J^\alpha_{\sqrt{ab}-}(fg)(a) + J^\alpha_{\sqrt{ab}+}(fg)(b)\right] \right| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln \frac{s}{a}\right)^{\alpha-1} |g(s)| \frac{ds}{s}\right)\left|f'(t)\right| dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s}\right)^{\alpha-1} |g(s)| \frac{ds}{s}\right)\left|f'(t)\right| dt \end{array} \right] \\ \leq\ & \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s}\right)\left|f'(t)\right| dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s}\right)\left|f'(t)\right| dt \end{array} \right]. \end{aligned}$$

Setting $t = a^{1-u} b^u$ and $dt = a^{1-u} b^u \ln\left(\frac{b}{a}\right) du$ gives

$$\begin{aligned} &\left| f\left(\sqrt{ab}\right)\left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b)\right] - \left[J^\alpha_{\sqrt{ab}-}(fg)(a) + J^\alpha_{\sqrt{ab}+}(fg)(b)\right] \right| \\ \leq\ & \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u} \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right) du \\ + \int_{\frac{1}{2}}^1 \left(\int_{a^{1-u}b^u}^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right) du \end{array} \right] \\ =\ & \frac{\|g\|_\infty \ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u} \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right) du \\ + \int_{\frac{1}{2}}^1 \left(\int_{a^{1-u}b^u}^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right) du \end{array} \right]. \end{aligned}$$

Using power mean inequality we have

$$
\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\
\leq\;& \frac{\|g\|_{\infty}\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}
\left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}} \\
\times\left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\
+\left[\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}} \\
\times\left[\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}
\end{array}\right] \\
=\;& \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}(\alpha+1)^{1-\frac{1}{q}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c}
\left[\int_0^{\frac{1}{2}}u^{\alpha}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\
+\left[\int_{\frac{1}{2}}^1(1-u)^{\alpha}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}
\end{array}\right].
\end{aligned} \tag{2.3}
$$

Since $|f'|^q$ is quasi-geometrically convex on $[a,b]$, we know that for $u\in[0,1]$

$$\left|f'(a^{1-u}b^u)\right|^q\leq\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}. \tag{2.4}$$

If we use (2.4) in (2.3), we have

$$
\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\
\leq\;& \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}(\alpha+1)^{1-\frac{1}{q}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c}
\left[\int_0^{\frac{1}{2}}u^{\alpha}\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\
+\left[\int_{\frac{1}{2}}^1(1-u)^{\alpha}\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}
\end{array}\right] \\
=\;& \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}(\alpha+1)^{1-\frac{1}{q}}\Gamma(\alpha+1)}\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}} \\
&\times\left[\left[\int_0^{\frac{1}{2}}u^{\alpha}\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(1-u)^{\alpha}\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}\right] \\
=\;& \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}(\alpha+1)^{1-\frac{1}{q}}\Gamma(\alpha+1)}\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}} \\
&\times\left[\left[\int_0^{\frac{1}{2}}u^{\alpha}\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(1-u)^{\alpha}\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}\right].
\end{aligned}
$$

This completes the proof. □

Corollary 2. *In Theorem 6:*

(1) If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer inequality for quasi-geometrically convex functions which is related to the left-hand side of (1.3):

$$\left|f\left(\sqrt{ab}\right)\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right|\leq\frac{\|g\|_{\infty}\ln^2\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}}C_2(1)\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}},$$

(2) If we take $g(x)=1$ we have the following Hermite-Hadamard inequality for quasi-geometrically convex functions in fractional integral forms which is related to the left-hand side of (1.4):

$$\left|f\left(\sqrt{ab}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)+J^{\alpha}_{\sqrt{ab}+}f(b)\right]\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2^{2-\frac{\alpha+1}{q}}(\alpha+1)^{1-\frac{1}{q}}}C_2(\alpha)\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}},$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for quasi-geometrically convex functions:

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right| \le \frac{\ln\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}} C_2(1) \left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}}.$$

Theorem 7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $f' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|f'|^q$, $q > 1$, is quasi-geometrically convex on $[a,b]$, $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals holds:*

$$\left| f\left(\sqrt{ab}\right) \left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b) \right] - \left[J^\alpha_{\sqrt{ab}-} (fg)(a) + J^\alpha_{\sqrt{ab}+} (fg)(b) \right] \right| \le \frac{\|g\|_\infty a \ln^{\alpha+1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}} (\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}} \Gamma(\alpha+1)}$$
$$\times \left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}} \left[\left[\left(\frac{b}{a}\right)^{\frac{q}{2}} - 1 \right]^{\frac{1}{q}} + \left[\left(\frac{b}{a}\right)^q - \left(\frac{b}{a}\right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \right],$$

with $\alpha > 0$ and $1/p + 1/q = 1$.

Proof. Using Lemma 1, Hölder's inequality and (2.4), setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u \ln\left(\frac{b}{a}\right) du$, we have

$$\left| f\left(\sqrt{ab}\right) \left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b) \right] - \left[J^\alpha_{\sqrt{ab}-} (fg)(a) + J^\alpha_{\sqrt{ab}+} (fg)(b) \right] \right|$$
$$\le \frac{1}{\Gamma(\alpha)} \left[\begin{array}{c} \int_a^{\sqrt{ab}} \left(\int_a^t \left(\ln \frac{s}{a}\right)^{\alpha-1} |g(s)| \frac{ds}{s} \right) |f'(t)| dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s}\right)^{\alpha-1} |g(s)| \frac{ds}{s} \right) |f'(t)| dt \end{array} \right]$$
$$\le \frac{\|g\|_\infty}{\Gamma(\alpha)} \left[\begin{array}{c} \int_a^{\sqrt{ab}} \left(\int_a^t \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s} \right) |f'(t)| dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s} \right) |f'(t)| dt \end{array} \right]$$
$$= \frac{\|g\|_\infty}{\Gamma(\alpha)} \left[\begin{array}{c} \int_0^{\frac{1}{2}} \left(\int_a^{a^{1-u}b^u} \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s} \right) \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) \ln\left(\frac{b}{a}\right) du \\ + \int_{\frac{1}{2}}^1 \left(\int_{a^{1-u}b^u}^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s} \right) \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) \ln\left(\frac{b}{a}\right) du \end{array} \right]$$
$$= \frac{\|g\|_\infty \ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)} \left[\begin{array}{c} \int_0^{\frac{1}{2}} \left(\int_a^{a^{1-u}b^u} \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s} \right) \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) du \\ + \int_{\frac{1}{2}}^1 \left(\int_{a^{1-u}b^u}^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s} \right) \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) du \end{array} \right]$$
$$\le \frac{\|g\|_\infty \ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)} \left[\begin{array}{c} \left(\int_0^{\frac{1}{2}} \left(\int_a^{a^{1-u}b^u} \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s} \right)^p du \right)^{\frac{1}{p}} \\ \times \left(\int_0^{\frac{1}{2}} \left|f'\left(a^{1-u}b^u\right)\right|^q \left(a^{1-u}b^u\right)^q du \right)^{\frac{1}{q}} \\ + \left(\int_{\frac{1}{2}}^1 \left(\int_{a^{1-u}b^u}^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s} \right)^p du \right)^{\frac{1}{p}} \\ \times \left(\int_{\frac{1}{2}}^1 \left|f'\left(a^{1-u}b^u\right)\right|^q \left(a^{1-u}b^u\right)^q du \right)^{\frac{1}{q}} \end{array} \right]$$
$$= \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}} (\alpha p+1)^{\frac{1}{p}} \Gamma(\alpha+1)} \left[\begin{array}{c} \left(\int_0^{\frac{1}{2}} \left|f'\left(a^{1-u}b^u\right)\right|^q \left(a^{1-u}b^u\right)^q du \right)^{\frac{1}{q}} \\ + \left(\int_{\frac{1}{2}}^1 \left|f'\left(a^{1-u}b^u\right)\right|^q \left(a^{1-u}b^u\right)^q du \right)^{\frac{1}{q}} \end{array} \right]$$
$$\le \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}} (\alpha p+1)^{\frac{1}{p}} \Gamma(\alpha+1)} \left[\begin{array}{c} \left(\int_0^{\frac{1}{2}} \sup\left\{|f'(a)|^q, |f'(b)|^q\right\} \left(a^{1-u}b^u\right)^q du \right)^{\frac{1}{q}} \\ + \left(\int_{\frac{1}{2}}^1 \sup\left\{|f'(a)|^q, |f'(b)|^q\right\} \left(a^{1-u}b^u\right)^q du \right)^{\frac{1}{q}} \end{array} \right]$$
$$= \frac{\|g\|_\infty a \ln^{\alpha+1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}} (\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}} \Gamma(\alpha+1)} \left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}}$$
$$\times \left[\left[\left(\frac{b}{a}\right)^{\frac{q}{2}} - 1 \right]^{\frac{1}{q}} + \left[\left(\frac{b}{a}\right)^q - \left(\frac{b}{a}\right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \right].$$

This completes the proof. □

Corollary 3. *In Theorem 7;*

(1) If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer inequality for quasi-geometrically convex functions which is related to the left-hand side of (1.3):

$$\left| f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \right| \leq \frac{\|g\|_\infty a \ln^{2-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{p+1}{p}} (p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}}$$
$$\times \left[\left[\left(\frac{b}{a}\right)^{\frac{q}{2}} - 1 \right]^{\frac{1}{q}} + \left[\left(\frac{b}{a}\right)^{q} - \left(\frac{b}{a}\right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \right],$$

(2) If we take $g(x)=1$ we have the following Hermite-Hadamard inequality for quasi-geometrically convex functions in fractional integral forms which is related to the left-hand side of (1.4):

$$\left| f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}} \left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b) \right] \right| \leq \frac{a \ln^{1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}+1-\alpha} (\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}}$$
$$\times \left[\left[\left(\frac{b}{a}\right)^{\frac{q}{2}} - 1 \right]^{\frac{1}{q}} + \left[\left(\frac{b}{a}\right)^{q} - \left(\frac{b}{a}\right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \right],$$

(3) If we take $\alpha=1$ and $g(x)=1$ we have the following Hermite-Hadamard inequality for quasi-geometrically convex functions:

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln\frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right| \leq \frac{a \ln^{1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{p+1}{p}} (p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}}$$
$$\times \left[\left[\left(\frac{b}{a}\right)^{\frac{q}{2}} - 1 \right]^{\frac{1}{q}} + \left[\left(\frac{b}{a}\right)^{q} - \left(\frac{b}{a}\right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \right].$$

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