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Risk Measures of the ERNB Distribution Generated by G-NB Family

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Abstract
This paper provides VaR and CVaR risk measures, calculated for the Erlang-Negative Binomial (ERNB) distribution. The Erlang and negative binomial distributions are given and then ERNB distribution is obtained using a family of univariate distributions which is called G-Negative Binomial (G-NB). It is defined as compounding the negative binomial distribution (NB) with any continuous distribution (G). Here, we use the ERNB distribution obtained as taking Erlang distribution instead of G. In this paper, we focus on the estimation of VaR and CVaR risk measures for this distribution in closed form and the explicit expressions are also presented for some parameter values. The results are portrayed in the figures. In additionally, numerical examples are given to illustrate changing of the risk measure according to some parameters on a real data set of automobile insurance policies.

Keywords: *Erlang distiribution; the Erlang-Negative Binomial distribution; Value at Risk; Conditional Value at Risk; G-NB family.*

AMS Subject Classification (2010): *Primary: 62H10 ; Secondary: 62P05; 91B30.*

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1. Introduction

Risk measures are frequently used in many problems in order to determine whether the risk is acceptable or not. These measures have a lot of applications in real life, such as in the risk theory, insurance and actuarial fields. For instance, the use of the risk measures in actuarial science helps to calculate important quantities such as premiums, retention and solvency. Value at Risk (VaR) and Conditional Value at Risk (CVaR) which we consider here, are well known risk measures in literature.

VaR is used to estimate the loss amount for a given time interval at a specific confidence level [1]. For continuous loss distributions, the conditional value-at-risk (CVaR) is used to quantify the expected losses that exceed the VaR. CVaR is also known as mean excess loss, mean shortfall, tail VaR, average value at risk or expected shortfall [2].

Although VaR is a popular risk measure, it is difficult to work with when losses are not "normally" distributed, which in fact is often the case, because loss distributions tend to exhibit " fat tails" or emprical discreteness [3]. Furthermore, it doesn't satisfy the properties required to be a coherent risk in the sense of Artzner et al.[4]. VaR is coherent only for normal distributions. Hence, the use of CVaR is recommended as an alternative risk measure to VaR in the literature since it has better properties than VaR such as subadditivity and convexity [3-5]. It has been proved by Pflug [6] and Artzner et al. [4] that CVaR is a coherent risk measure.

The coherent property is important especially in optimization problems because it facilitates mathematical operations. CVaR and its minimization formula were first developed by Rockafellar and Uryasev [3]. There are also many studies related to CVaR in actuarial fields such as Krokhmal et al.[7], Jiménez et al. [8], Embrechts et al. [9], and Rau-Bredow [10].

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It is important to describe individual and collective claim distiributions in risk and actuarial fields. Gamma, lognormal, Weibull, or Pareto distributions are often used for modeling the claim data [11]. The Erlang distribution is widely used in modeling the distribution of claims in many applications such as in insurance and risk [12-15]. However, some problems can be seen in fitting the insurance loss data such as asymmetry and skewness. To overcome these problems, some mixed models, especially Erlang distribution with gamma kernel density estimators are used in the literature [16-18].

One of the other problem encountered in the literature is the question of overdispersion. The Poisson distribution is used for counts events such that it usually fits to the data in various fields. When a variance is larger than the mean it is called overdispersion, the negative binomial (NB) is employed as an alternative to count data with overdispersion instead of the Poisson distribution. The NB distribution was first introduced by Pascal [19]. The NB distribution was used by Student [20] as an alternative to the Poisson distribution. Therefore, in the literature lots of applications are performed using the NB distribution in combination with various distributions, such as the negative binomial-inverse Gaussian distributions [21], the negative binomial-Beta Exponential distribution [22], the BuRR XII- negative binomial distribution [23] and the Erlang-negative binomial distribution [24].

In some fields such as automobile insurance, it can be encountered with count data that has a large number of zeros in Poisson distribution which can also be referred to as heavy- tailed losses. In other words, losses may have a distribution with heavy tails. It is recommended to use the ERNB distribution for heavy- tailed losses [24].

In this paper, we concentrated on the Erlang-negative binomial distribution generated using a family of univariate distributions (G-NB). This distribution occures with compounding the negative binomial distribution with any continuos cdf G(x) [25]. An application on the life time data has been given in Ramos et al. [23] by combining Burr XII and Negative binomial distributions. Here, the Erlang-negative binomial (ERNB) distribution is obtained by this approach.

So, we present a framework to establish VaR and CVaR in closed form when the loss distribution exhibits ERNB. The results are compared for various parameter values and illustrated in figures. As a numerical example, we use a real automobile insurance data fitted with the ERNB introduced by Kongrod et al. [24]. To our knowledge, this is the first time that ERNB distribution has been used to calculate CVaR.

The rest of paper is organized as follows. Section 2 presents the Erlang-Negative Binomial distribution generated by using a the G-Negative Binomial (G-NB) family. In Section 3, we give closed forms of the results VaR and CVaR based on the ERNB distribution, and Section 4 includes an example of calculation VaR and CVaR for a real data. Finally, in Section 5, the conclusions of the study are presented.

2. The Erlang-Negative Binomial Distribution

The Erlang distribution was firstly introduced by Erlang [26]. The probability density function (pdf) and cumulative distribution function (cdf) of the Erlang distribution are given by

$$g\left(x;c,k\right)=\frac{c^{k}x^{k-1}e^{-cx}}{(k-1)!},x>0 \tag{2.1}$$

and

$$G\left(x;c,k\right)=\frac{\gamma\left(k,cx\right)}{(k-1)!}\quad or\quad G\left(x;c,k\right)=1-e^{-cx}\sum_{n=0}^{k-1}\frac{1}{n!}(cx)^{n} \tag{2.2}$$

respectively, where k is an integer value and it is called the shape parameter, and $c>0$ is called the rate parameter. $\gamma\left(.\right)$ is the lower incomplete gamma function. Figure 1 shows the graphs of the pdf and cdf of the Erlang distribution with the various values of parameters.

The negative binomial distribution is often used for the counts of event in risk analysis. In applications, overdispersion problem occurs when the variance is larger than the mean. In such cases, the NB distribution can be used as an alternative to the Poisson distribution.

The pdf and cdf of negative binomial distribution are given respectively as

$$f\left(x,s,\beta\right)=\left(\begin{array}{c}s+x-1\\x\end{array}\right)\beta^{s}(1-\beta)^{x},\quad x=0,1,2,\ldots,s>0,\ \ 0<\beta<1 \tag{2.3}$$

and

$$F\left(x,s,\beta\right)=\sum_{n=0}^{x}\left(\begin{array}{c} n+x-1 \\ x \end{array}\right)\beta^{s}(1-\beta)^{n} \tag{2.4}$$

where β is the probability of success and s is the number of successes.

A new family of univariate distributions was generated by Percontini *et al*. [25]. This distribution is occured by compounding the negative binomial distribution with any continuos cdf $G(x)$. It is called as the *G-Negative Binomial* (G-NB) family of distributions with pdf $f(x)$ and cdf $F(x)$ and respectively is defined as

$$f\left(x\right)=\frac{s\beta}{\left[(1-\beta)^{-s}-1\right]}g\left(x\right)\left\{1-\beta\left[1-G\left(x\right)\right]\right\}^{-s-1} \tag{2.5}$$

and

$$F\left(x\right)=\frac{(1-\beta)^{-s}-\left\{1-\beta\left[1-G\left(x\right)\right]\right\}^{-s}}{\left[(1-\beta)^{-s}-1\right]}. \tag{2.6}$$

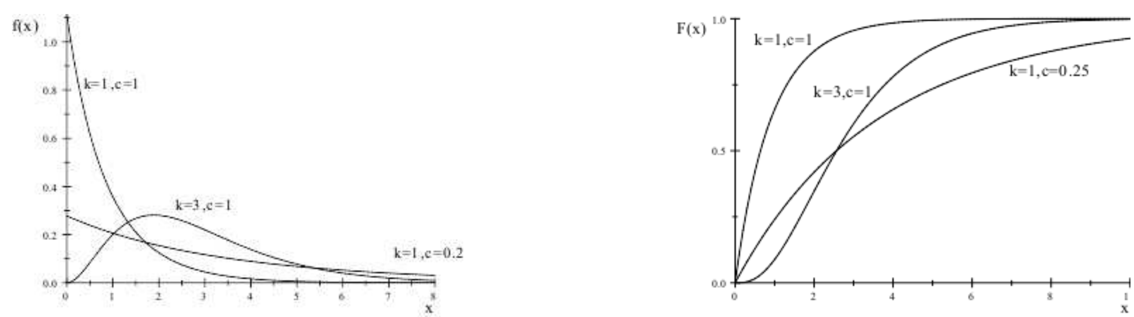
We obtain the $ERNB\left(s,\ \beta,\ c,\ k\right)$ distribution by inserting equations (2.2) into (2.5) and (2.6). Then, the pdf and cdf of the generated distribution (for $x>\ 0$) can be given by

$$f\left(x\right)=\frac{s\beta}{\left[(1-\beta)^{-s}-1\right]}\frac{c^{k}x^{k-1}e^{-cx}}{(k-1)!}\left\{1-\beta\left[\sum_{n=0}^{k-1}\frac{1}{n!}e^{-cx}(cx)^{n}\right]\right\}^{-s-1} \tag{2.7}$$

and

$$F\left(x\right)=\frac{(1-\beta)^{-s}-\left\{1-\beta\left[\sum_{n=0}^{k-1}\frac{1}{n!}e^{-cx}(cx)^{n}\right]\right\}^{-s}}{\left[(1-\beta)^{-s}-1\right]} \tag{2.8}$$

respectively, where $s>0$, $c>0$, k is an integer value and $\beta\in(0,\ 1)$. The graphs of the pdf and cdf of the ERNB with the various values of parameters are shown in Figure 2.



(a) **(b)**

Figure 1. For $s=1$, $\beta=0.1$ (a) Pdf of the Erlang Distribution (b) Cdf of the Erlang Distribution

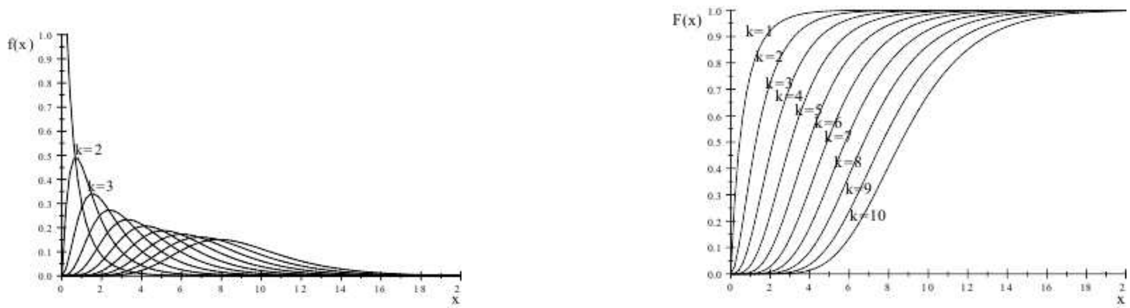
3. Risk Measures of the ERNB Distribution

VaR and $CVaR$ measures for a risk X with a probability level $p\in\ (0,1)$ are respectively defined as

$$VaR_{p}(X)=\inf\left\{x:F_{X}\left(x\right)\geq p\right\}=F_{X}^{-1}(p), \tag{3.1}$$

$$CVaR_{p}\left(X\right)=E\left[X-VaR_{p}(X)|X>VaR_{p}(X)\right]=CTE_{p}\left(X\right)-\ VaR_{p}(X) \tag{3.2}$$

where CTE (conditional tail expectation) is defined as $CTE_{p}\left(X\right)=E\left[X|X>VaR_{p}(X)\right]$. If F_X is continuous then it can be written $CTE_{p}\left(X\right)=\frac{1}{1-p}\int_{p}^{1}VaR_{\xi}(X)d\xi$. (Denuit et al.[2]).



(a) (b)

Figure 2. For $s = 10$, $c = 1$ and $\beta = 0.1$ (a) Pdf of the ERNB Distribution (b) Cdf of the ERNB Distribution

In this study, VaR and $CVaR$ measures are obtained for the ERNB distribution defined by (2.8) equation by using (3.1) and (3.2). We give the results for Case 1 $(s = 1, k = 1)$ and Case 2 $(s = 2,\ \ k = 1)$ below. Also in the application section, a numerical example is given for real data which is modeled with the ERNB by Kongrod et al. [24].

Case 1: $(s = 1, k = 1)$

The cdf of the ERNB is given by

$$F(x) = \left(\frac{1 - e^{-cx}}{1 - \beta e^{-cx}}\right) \tag{3.3}$$

Then, we can obtained VaR and CVaR respectively as follows

$$VaR_p(X) = -\frac{1}{c} ln\left(\frac{1-p}{1-p\beta}\right) \quad and \ CVaR_p(X) = \frac{(1-\beta)}{c\beta(1-p)} ln \frac{(p\beta - 1)}{(\beta - 1)} \tag{3.4}$$

Case 2: $(s = 2, k = 1)$

The cdf of the ERNB is given by

$$F(x) = \frac{\beta - \beta e^{-2cx} + 2e^{-cx} - 2}{(\beta - 2)(\beta e^{-cx} - 1)^2} \tag{3.5}$$

Then, we can obtained VaR and CVaR respectively as follows

$$VaR_p(X) \ = \ -\frac{1}{c} ln\left(-\frac{1}{\beta}\left(\sqrt{\frac{(\beta-1)^2}{p\beta^2 - 2p\beta + 1}} - 1\right)\right) \tag{3.6}$$

$$CVaR_p(X) \ = \frac{1}{1-p}\int_p^1 -\frac{1}{c} ln\left(-\frac{1}{\beta}\left(\sqrt{\frac{(\beta-1)^2}{p\beta^2 - 2p\beta + 1}} - 1\right)\right) dp \tag{3.7}$$

Numerical results of Case 1 and Case 2 are given in Table 1. Figures 3 and 4 show the changing of the VaR and CVaR of the ERNB for different values of β and p.

For other cases, we can use the G-NB quantile function approach expressed by Percontini et al [25] (for $0 < p < 1$) as

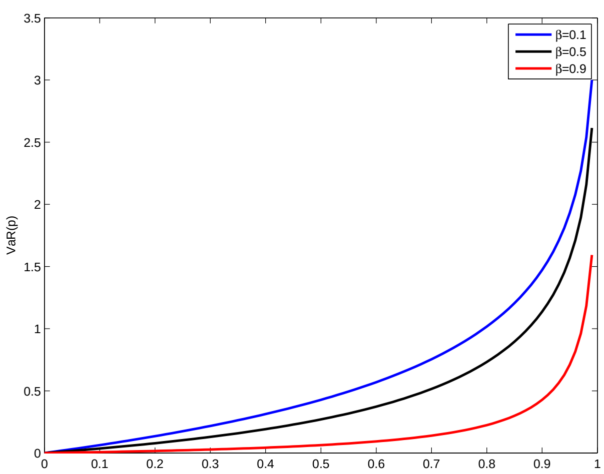
$$Q(p) = Q_G\left\{1 - \frac{1}{\beta}\left[1 - (1-\beta)(1 - p[1 - (1-\beta)^s])^{-\frac{1}{s}}\right]\right\}$$

where $Q_G(p)$ is quantile function. Quantile function can be considered as Value at Risk (VaR) in actuarial area. In this approach, firstly, quantile function of the Erlang distribution is found by solving $Q_{ER} = inf\left\{x : 1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-cx}(cx)^n > p\right\}$. Then, quantile values of the ERNB distribution for given parameter values can be founded as numerically, writing $Q_G(p)$ place of p. The VaR, CTE and CVaR values of the ERNB with various values of parameters are given in Table 1. We display the figures of VaR and CVaR for various values of parameters in Figures 3 and 4, respectively.

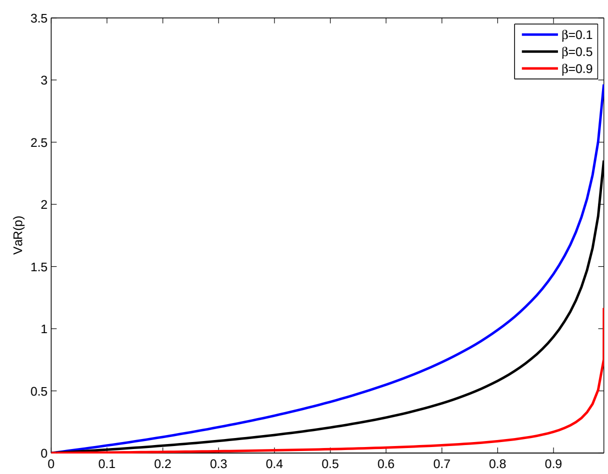
Figures 3 and 4 show that VaR and CVaR decrease as β increases in both cases. The losses obtained for CVaR in each case are greater than the losses of VaR. VaR and CVaR values are increasing with respect to p but the increase in CVaR is slower than VaR. It can even be said that it is almost constant for very small β values. On the other hand, it is seen from Table 1 that even a unit increase in the s parameter causes significant decreases in VaR and CVaR values for each p value. This decrease is faster for high confidence levels $(p > 0.6)$, especially for CVaR.

Table 1: The VaR, CTE and CVaR values of ERNB for various values of (s, k, β, c, p)

s	k	β	c	p	$VaR_p(X)$	$CTE_p(X)^*$	$CVaR_p(X)$
1	1	0.1	1.5	0.3	0.2175	0.8595	0.6420
				0.6	0.5697	1.2219	0.6522
				0.9	1.4722	2.1352	0.6630
		0.5	1.5	0.3	0.1294	0.6348	0.5054
				0.6	0.3730	0.9339	0.5609
				0.9	1.1365	1.7719	0.6354
		0.9	1.5	0.3	0.0279	0.2383	0.2104
				0.6	0.0932	0.3758	0.2826
				0.9	0.4279	0.9034	0.4755
2	1	0.1	1.5	0.3	0.2080	0.8373	0.6293
				0.6	0.5496	1.1941	0.6445
				0.9	1.4404	2.1013	0.6609
		0.5	1.5	0.3	0.0974	0.5185	0.4211
				0.6	0.2855	0.7730	0.4875
				0.9	0.9353	1.5383	0.6030
		0.9	1.5	0.3	0.0144	0.1053	0.0909
				0.6	0.0436	0.1652	0.1216
				0.9	0.1703	0.4057	0.2354



(a)

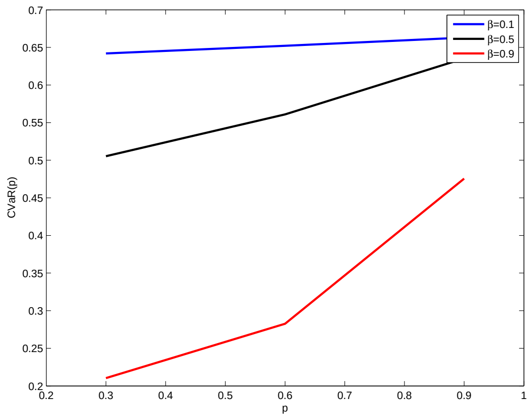


(b)

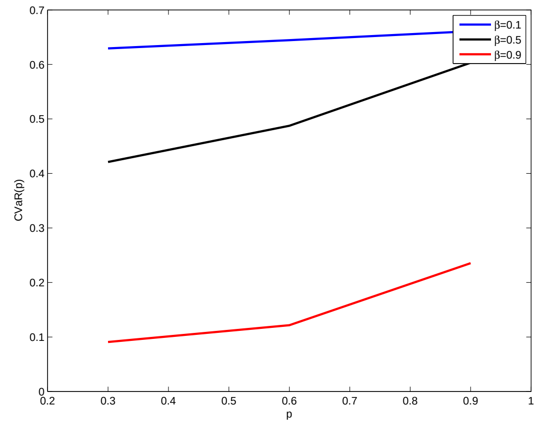
Figure 3. The VaR of ERNB for various values of β and p (a) $s = 1$, $k = 1$ (b) $s = 2$, $k = 1$

4. Numerical Example

We use the data set includes in 9461 automobile insurance policies consist of the accident numbers of each policy. Kongrod et al [24] applied this data set to fit with the Poission, the NB and the ERNB distributions, and the ERNB distribution is shown to be best fit among three distributions. Parameter estimations are obtained as $\hat{s} = 0.311,\ \hat{k} = 3.428,\ \hat{\beta} = e^{-0.214} = 0.8073,\ \ \hat{c} = 5.127$ by Kongrod et al [24].



(a)



(b)

Figure 4. The VaR of ERNB for various values of β and p (a) $s = 1$, $k = 1$ (b) $s = 2$, $k = 1$

In this study VaR and CVaR are calculated using the G-NB approach for the values of parameter estimations obtained by Kongrod et al. [24]. The loss of calculation VaR, CTE and CVaR for number of accidents of each policy in the automobile insurance are presented in Table 2 under the following confidence levels: (p: 10%-80%). The changing of VaR, CTE and CVaR with respect to *p* are shown in Figure 5. As can be noted, while CVaR losses up to about confidence level of 35% are greater than the losses of VaR, it is seen that CVaR losses fall to under the losses of VaR after confidence level of 35%. In addition to, for the fixed values of parameters, whereas the losses of VaR are increasing with respect to confidence level of p, the losses of CVaR are decreasing.

Table 2: VaR , CTE and CVaR values by G-NB quantile approach for numerical example

$$(\hat{s} = 0.311,\ \hat{k} = 3.428,\ \hat{\beta} = e^{-0.214} = 0.8073,\ \ \hat{c} = 5.127)$$

p	$VaR_p\,(X)$	$CTE_p\,(X)$	$CVaR_p\,(X)$
0.1	0.1356	0.4468	0.3112
0.2	0.1879	0.4823	0.2945
0.3	0.2352	0.5210	0.2859
0.4	0.2840	0.5646	0.2807
0.5	0.3380	0.6155	0.2775
0.6	0.4018	0.6771	0.2754
0.7	0.4826	0.7561	0.2735
0.8	0.5957	0.8666	0.2710

5. Discussion and Conclusion

We have presented the estimation VaR and CVaR of common risk measures for the ERNB distribution generated using the G-Negative Binomial (G-NB) family approach in the case of heavy-tailed losses. The formulas obtained for calculating VaR and CVaR have been provided in closed form and presented the explicit expressions for the fixed values of parameters. Some numerical computations for risk measures VaR and CVaR have been applied to real insurance data fitted with the ERNB.

The recommended model to take actuarial decisions in cases of losses with heavy tail distributed or overdispersion provides the advantage. As future work, the VaR and CVaR formulas for ERNB obtained in this study can be also improved for other parameter values, and significant results can be achieved by taking into account of the formulas in the popular actuarial issues such as the non-life insurance and ruin probability calculations.

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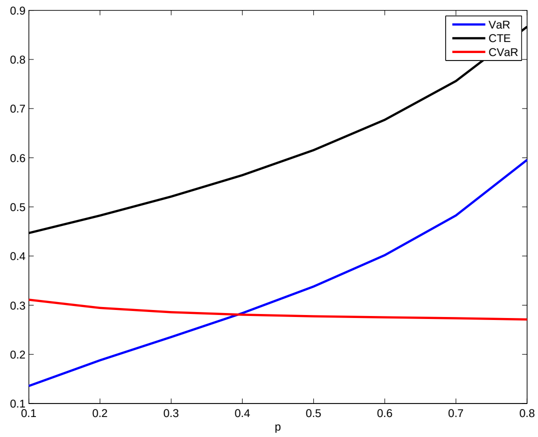

Figure 5. The VaR, CTE and CVaR for 9461 automobile insurance policies

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