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Extensions of Morphic Quasi-morphic and Centrally Morphic Rings

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Abstract
In the present paper, we define the ring

$$T = [R; I, \sigma, n] := \Big\{ a_0 + a_1 x + \ldots + a_n x^n \in \frac{(R, I)[x; \sigma]}{(x^{n+1})} : a_0 \in R, a_i \in I \text{ for } i = 1, \ldots, n \Big\},$$

which is a subring of $R[x;\sigma]/(x^{n+1})$. It is proved that; If R is a unit regular ring, each $\alpha \in T$ is equivalent to an element $e_0 + e_1 x + \cdots + e_n x^n$, where $e_0, e_1, ..., e_n$ is a sequence of orthogonal idempotents such that $e_0 \in R$, $e_1, ..., e_n \in I$ and $n \geq 1$. As an application of this, it has shown that;

(1) The ring $[R; I, \sigma, n]$ is left morphic.

(2) $(R, I)(x)/(x^{n+1})$ is left centrally morphic for each $n \geq 0$.

Also, we prove that the ring $(R, I)(x)/(x^{n+1})$ is left quasi-morphic.

Keywords: *Morphic rings; quasi-morphic rings; unit (strongly) regular rings; centrally morphic rings*

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1. Introduction

Throughout this paper, all rings are associative with identity. For $a, b \in R$, we say that a is *equivalent* to b, if $b = uav$ for some units u and v in R. Let R be a ring. For any element a in R, the left (respectively, right) annihilator of a in R is denoted by $\mathbf{l}_R(a)$ (respectively, $\mathbf{r}_R(a)$), and it is well known that $R/\mathbf{l}_R(a) \cong Ra$ as left R-modules.

If $R/Ra \cong \mathbf{l}_R(a)$, then a is called a *left morphic* element (see [10]). Equivalently, $a \in R$ is left morphic if and only if there exists $b \in R$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$ (see [10, Lemma 1]). A ring R is called *left morphic* if every element of R is left morphic. An element a in R is *unit regular* if there exists $u \in U(R)$ such that $a = aua$, where $U(R)$ denotes the group of units of R. The ring itself is unit regular if all of its elements are unit regular. R is *unit-strongly regular ring*, every element r of R there exists a unit element $u \in R$, $r = r^2u$. Ehrlich [3], has shown that a ring is unit regular if and only if it is (von Neumann) regular and left morphic. In [7] Lee and Zhou studied the relationships between these properties for rings of the form S/I, where S is either a polynomial ring $R[x]$ or a skew polynomial ring $R[x;\sigma]$, and where I is an ideal of the form (x^n). Thus by [7, Theorem 2] if R is unit regular, and if the endomorphism $\sigma : R \to R$ is onto and fixes all idempotents of R, then all such rings S/I are left morphic. This had previously been known only when R is strongly regular by [2].

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Let I be an ideal of R and $\sigma : R \to R$ be a ring endomorphism. Then define ring T as following:

$$T = [R; I, \sigma, n] := \Big\{ a_0 + a_1 x + \ldots + a_n x^n \in \frac{(R, I)[x; \sigma]}{(x^{n+1})} : a_0 \in R, a_i \in I \text{ for } i = 1, \ldots, n \Big\}.$$

It is clear that T is a ring (a subring of $R[x; \sigma]/(x^{n+1})$). The ring S below is the special case of T where $\sigma = 1_R$. Let $\sigma : R \to R$ is a ring endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$. In section 2, it is shown that $[R; I, \sigma, n]$ is a left morphic ring for every $n \geq 0$. This result is a generalization of [8, Corollary 3].

The ring R is called *left centrally morphic* if, for each $a \in R$, there exists $b \in C(R)$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$, where $C(R)$ denotes the center of R (see [9, Section 5]). In [6], Huang and Chen considered the following situation: Let I be an ideal of a unit regular ring R and let

$$S := \frac{(R, I)[x]}{(x^{n+1})} = \Big\{ \sum_{i=0}^{n} a_i x_i : a_0 \in R, a_i \in I, i = 1, 2, \cdots, n \Big\}.$$

They showed in [6, Theorem 2.2] that every matrix ring over S is morphic. In [9, Section 5], Lee and Zhou proved that, for an integer $n \geq 1$, a ring R is strongly regular iff $R[x]/(x^{n+1})$ is left centrally morphic. Note that this result is a generalization of [7, Corollary 5] and [7, Theorem 12]. In section 3, we generalize this result for the ring S.

An element $a \in R$ is called *left quasi-morphic* if there exist $b, c \in R$ such that $Ra = l(b)$ and $Rb = l(a)$ (see [1]). This notion was introduced as a generalization of left morphic rings and regular rings. Also in [1], it was shown that left quasi morphic rings share a number of important properties with regular rings. In section 4, we used the technique which was a generalization of technique that Herbera used in [5]. By using that, we prove that the ring S is quasi morphic for each $n \geq 0$.

2. The ring $[R; I, \sigma, n]$ is left morphic

Proposition 1. *[8, Proposition 1] Let R be a semiprime ring and let σ be an endomorphism of R such that $\sigma(e) = e$ for all $e^2 = e \in R$. Then $e(\sigma^k(x) - x)(1 - e) = 0$ for all $x \in R$, all $e^2 = e \in R$ and positive integers k.*

The following Theorem is a generalization of [8, Theorem 2]. We prove it by the similar way.

Theorem 2. *Let I be an ideal of a unit regular ring R and let $\sigma : R \to R$ is a ring endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$. Then each $\alpha \in T$ is equivalent to $e_0 + e_1 x + \cdots + e_n x^n$, where $e_0, e_1, ..., e_n$ is a sequence of orthogonal idempotents such that $e_0 \in R$ and $e_1, ..., e_n \in I$ and $n \geq 1$.*

Proof. It is enough to prove the following claim:
Claim: For each integer k, there exists idempotents $e_0 \in R$, $e_1, \ldots, e_{k-1} \in I$, and $r_k, \ldots, r_n \in I$ such that up to equivalence

$$\alpha = e_0 + e_1 x + \cdots + e_{k-1} x^{k-1} + \sum_{j=k}^{n} r_j x^j, \quad (*)$$

where $e_i \in (1 - e_{i-1}) \ldots (1 - e_0) I (1 - e_0) \ldots (1 - e_{i-1})$ for $i = 1, \ldots, k-1$ and $r_j \in (1 - e_{k-1}) \ldots (1 - e_0) I (1 - e_0) \ldots (1 - e_{k-1})$ for $j = k, \ldots, n$. When we take $k = n$, theorem will be proved. In this case we have that

$$\alpha = e_0 + e_1 x + \cdots + e_{k-1} x^{k-1} + r_n x^n,$$

where $e_i \in (1-e_{i-1}) \ldots (1-e_0) I (1-e_0) \ldots (1-e_{i-1})$ for $i = 1, \ldots, n-1$ and $r_n \in hIh$ with $h := (1-e_0) \ldots (1-e_{n-1})$. It is known hRh unit-regular by [4, Corollary 4.7], so there exists a unit $u \in hRh$ with inverse v and an idempotent $e_n \in hRh$ such that $r_n = ue_n$. We have $e_n = vr_n \in hIh$, because $r_n \in hIh$,. Clearly, $(e_0 + \cdots + e_{n-1}) + v$ is a unit in R and

$$(e_0 + \cdots + e_{n-1} + v)\alpha = e_0 + e_1 x + \cdots + e_{n-1} x^{n-1} + e_n x^n.$$

Proof of Claim : We prove it by induction on k.

Let $k = 1$ and $\alpha = r_0 + r_1 x + \cdots + r_n x^n \in T$. Every r_0 can be written as a product of unit and an idempotent because R is unit regular. Hence, up to equivalence, left multiplying α by a suitable unit of R, we can assume that $r_0 = e_0$ is an idempotent. Because

$$(1 - (1 - e_0) r_1 x) \alpha (1 - r_1 x) = e_0 + (1 - e_0) r_1 (1 - e_0) x + \ldots$$

where both $1-(1-e_0)r_1x$ and $1-r_1x$ are units of T, we can further assume that $r_1 \in (1-e_0)I(1-e_0)$. Now

$$(1-(1-e_0)r_2x^2)\alpha(1-r_2x^2) = e_0 + r_1x + (1-e_0)r_2(1-e_0)x^2 + \dots$$

where both $1-(1-e_0)r_2x^2$ and $1-r_2x^2$ are units of T, so we can assume that $r_2 \in (1-e_0)I(1-e_0)$. A simple induction shows that

$$\alpha = e_0 + r_1x + r_2x^2 + \cdots + r_nx^n \ \ r_i \in (1-e_0)I(1-e_0) \ \ for \ \ i = 1, \dots, n.$$

So the case $k = 1$ is proved.

Assume that $(*)$ holds for a fixed integer k with $1 < k < n$. It is clear that $e_0, \dots, e_{k-1}$ are orthogonal idempotents. Set
$f_{k-1} := (1-e_0)\dots(1-e_{k-1}) \in I$ and $g_{k-1} := e_0+\cdots+e_{k-1} \in R$. Then f_{k-1} and g_{k-1} are orthogonal idempotents and $f_{k-1} + g_{k-1} = 1$. Because $f_{k-1}Rf_{k-1}$ is unit regular by [4, Corollary 4.7], write $r_k = ue_k$ where e_k is an idempotent element in $f_{k-1}Rf_{k-1}$ and u is an unit element of $f_{k-1}Rf_{k-1}$ with inverse v. Then $e_k = vr_k \in f_{k-1}If_{k-1}$, since $r_k \in f_{k-1}If_{k-1}$. Then $g_{k-1} + v$ is a unit of R with inverse $g_{k-1} + u$. Since

$$(g_{k-1} + v)\alpha = e_0 + e_1x + \cdots + e_kx^k + \sum_{j=k+1}^{n} vr_jx^j,$$

up to equivalence, we can assume that

$$\alpha = e_0 + e_1x + \cdots + e_kx^k + \sum_{j=k+1}^{n} r_jx^j,$$

where $e_k^2 = e_k \in f_{k-1}If_{k-1}$ and $r_j \in f_{k-1}If_{k-1}$ for $j = k+1, \dots, n$. Now

$$\alpha' := (1 - r_{k+1}x)\alpha = e_0 + e_1x + \cdots + e_kx^k + r_{k+1}(1-e_k)x^{k+1} + \sum_{j=k+2}^{n} r'_jx^j,$$

where $r_{k+1}, r'_{k+2}, \dots, r'_n \in f_{k-1}If_{k-1}$. Set $r'_{k+1} := r_{k+1}(1-e_k)$. Then compute,

$$(1-(1-e_k)r'_{k+1}x)\alpha'(1-r'_{k+1}x) = \sum_{i=0}^{k} e_ix^i + \sum_{j=k+1}^{n} r''_jx^j,$$

where

$$\begin{aligned} r''_{k+1} &= r'_{k+1} - e_k\sigma^k(r'_{k+1}) - (1-e_k)r'_{k+1}e_k \\ &= e_k(r'_{k+1} - \sigma^k(r'_{k+1})) + (1-e_k)r'_{k+1}(1-e_k) \\ &= e_k(r_{k+1} - \sigma^k(r_{k+1}))(1-e_k) + (1-e_k)r'_{k+1}(1-e_k) \\ &= (1-e_k)r'_{k+1}(1-e_k) \in (1-e_k)f_{k-1}If_{k-1}(1-e_k) \end{aligned}$$

since $e_k(r_{k+1} - \sigma^k(r_{k+1}))(1-e_k)$ by Proposition 1, and where all $r''_i \in f_{k-1}If_{k-1}$ for $i \geq k+2$.

We set $f_i := (1-e_0)\dots(1-e_i)$ for $i = 0, 1, \dots, k$. Up to equivalence we may assume that

$$\alpha = \sum_{i=0}^{k} e_ix^i + r_{k+1}x^{k+1} + \sum_{j=k+2}^{n} r_jx^j,$$

where $e_i = e_i^2 \in f_{i-1}If_{i-1}$ for $i = 1, \dots, k$ and where $r_{k+1} \in f_kIf_k$, $r_j \in f_{k-1}If_{k-1}$ $j = k+2, \dots, n$. Then compute

$$\begin{aligned} \alpha' &= (1 - r_{k+2}x^2)\alpha \\ &= \textstyle\sum_{i=0}^{k} e_ix^i + r_{k+1}x^{k+1} + \sum_{j=k+2}^{n} r'_jx^j, \end{aligned}$$

where $r'_j \in f_{i-1}If_{i-1}$ for $j > k+2$ and where $r'_{k+2} = r_{k+2}(1-e_k)$. Then compute

$$(1-(1-e_k)r'_{k+2}x^2)\alpha'(1-r'_{k+2}x^2) = \sum_{i=0}^{k} e_ix^i + r_{k+1}x^{k+1} + \sum_{j=k+2}^{n} r''_jx^j,$$

where

$$\begin{aligned} r''_{k+2} &= r'_{k+2} - e_k\sigma^k(r'_{k+2}) - (1-e_k)r'_{k+2}e_k \\ &= e_k(r'_{k+2} - \sigma^k(r'_{k+2})) + (1-e_k)r'_{k+2}(1-e_k) \\ &= e_k(r_{k+2} - \sigma^k(r_{k+2}))(1-e_k) + (1-e_k)r'_{k+2}(1-e_k) \\ &= (1-e_k)r'_{k+2}(1-e_k) \in (1-e_k)f_{k-1}If_{k-1}(1-e_k) = f_kIf_k \end{aligned}$$

since $e_k(r_{k+2} - \sigma^k(r_{k+2}))(1-e_k)$ by Proposition 1, and where all $r''_i \in f_{k-1}If_{k-1}$ for $i \geq k+3$. Repeating analogous arguments, up to equivalence we may assume that

$$\alpha = e_0 + e_1x + \cdots + e_kx^k + \sum_{j=k+1}^{n} r_jx^j,$$

where $r_j \in f_kIf_k$ for $j = k+1, \ldots, n$. So we complete the inductive step and we are done. □

Corollary 3. *Let I be an ideal of a unit regular ring R and let $\sigma : R \to R$ is a ring endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$. Then T is a left morphic ring for each $n \geq 0$.*

Proof. We will show arbitrary $\alpha \in T$ is left morphic in $T = [R; I, \sigma, n]$. Let $\beta = \sum_{i=0}^{n} b_ix^i \in T$, where $b_0 = (1-e_0)(1-e_1)\ldots(1-e_n) = 1 - e_0 - e_1 - \cdots - e_n$ and $b_i = e_{n-i}$ for $i = 1, \ldots, n$.
Claim : $T\alpha = \mathbf{l}_T\beta$ and $T\beta = \mathbf{l}_T\alpha$. By Theorem 2, α is equivalent to $\gamma := e_0 + e_1x + \cdots + e_nx^n$, where $e_0^2 = e_0 \in R$ and $e_i^2 = e_i \in (1-e_{i-1})\ldots(1-e_0)I(1-e_0)\ldots(1-e_{i-1})$ for $i = 1, \ldots, n$.

Given $\lambda = \sum_{i=0}^{n} r_ix^i \in T$ with $r_0 \in Re_0$ and $r_i \in I\sum_{j=0}^{i} e_j$, let $\gamma = \sum_{i=0}^{n} a_ix^i \in T$, where

$$\begin{aligned} a_0 &= r_0e_0 + r_1e_1 + \cdots + r_ne_n \\ a_1 &= r_1e_0 + r_2e_1 + \cdots + r_ne_{n-1} \\ &\vdots \\ a_i &= r_ie_0 + r_{i+1}e_1 + \cdots + r_ne_{n-i} \\ &\vdots \\ a_n &= r_ne_0 \end{aligned}.$$

Then $\lambda = \gamma\alpha \in T\alpha$.

For any $\omega = \sum_{i=0}^{n} a_ix^i \sum_{j=0}^{n} e_jx^j \in T\alpha$, the coefficient of x^k, is

$$\begin{aligned} &\textstyle\sum_{i=0}^{k} a_i\sigma^i(e_{k-i}) = \sum_{i=0}^{n} a_i(e_{k-i}) = a_0e_k + a_1e_{k-1} + \cdots + a_ke_0 \\ &= (a_0e_k + a_1e_{k-1} + \cdots + a_ke_0)(e_0 + e_1 + \cdots + e_k) \in I(e_0 + e_1 + \cdots + e_k). \end{aligned}$$

Hence,

$$\begin{aligned} T\alpha = \ &\{r_0 + r_1x + \cdots + r_{n-1}x^n : r_0 \in Re_0, r_1 \in I(e_0 + e_1), \ldots, \\ &r_n \in I(e_0 + e_1 + \cdots + e_n)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} T\beta = \ &\{r_0 + r_1x + \cdots + r_{n-1}x^n : r_0 \in R(1 - e_0 - e_1 - \cdots - e_n), \\ &r_1 \in I(1 - e_0 - e_1 - \cdots - e_{n-1}), \ldots, r_n \in I(1-e_0)\}. \end{aligned}$$

For any $\gamma = \sum_{i=0}^{n} a_ix^i \in T$, we have $\gamma \in \mathbf{l}_T(\alpha)$ if and only if

$$\begin{aligned} &a_0e_0 = 0, \\ &a_0e_1 + a_1\sigma(e_0) = 0, \\ &a_0e_2 + a_1\sigma(e_1) + a_2\sigma^2(e_0) = 0, \\ &\vdots \\ &a_0e_n + a_1\sigma(e_{n-1}) + \cdots + a_n\sigma^n(e_0) = 0, \end{aligned}$$

but since $\sigma(e) = 0$ for all idempotens, we have,

$$\begin{aligned} &a_0e_0 = 0, \\ &a_0e_1 + a_1e_0 = 0, \\ &a_0e_2 + a_1e_1 + a_2e_0 = 0, \\ &\vdots \\ &a_0e_n + a_1e_{n-1} + \cdots + a_ne_0 = 0, \end{aligned}$$

if and only if

$$\begin{aligned} a_0e_i &= 0 \ \ (0 \le i \le n), \\ a_1e_i &= 0 \ \ (0 \le i \le n-1), \\ &\vdots \\ a_je_i &= 0 \ \ (0 \le i \le n-j), \\ &\vdots \\ a_ne_0 &= 0; \end{aligned}$$

if and only if $a_0 \in R(1-\sum_{i=0}^{n} e_i)$ and $a_j \in I(1-\sum_{i=0}^{n-j} e_i)$ for any j with $1 \le j \le n$. Hence $\mathbf{l}_T(\alpha) = T\beta$. Using similar argument one can have $\mathbf{l}_T(\beta) = T\alpha$. So T is left morphic. □

Corollary 4. *[8, Corollary 3] If R is a unit regular ring and $\sigma : R \to R$ is a ring endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$. Then $R[x;\sigma]/(x^{n+1})$ is a left morphic ring for all $n \ge 0$.*

Proof. Let take $I = R$ in the previous corollary then the result follows. □

3. The ring $(R, I)(x)/(x^{n+1})$ is centrally morphic

Remark 5. *If R is a unit-regular ring and if $\alpha \in S = (R, I)(x)/(x^{n+1})$ where $n \ge 0$, then by Theorem 2 there exist a sequence of orthogonal idempotents $e_0, e_1, ..., e_n$,where $e_0 \in R$, $e_1, ..., e_n \in I$, and units u,v $\in U(S)$ such that $\alpha = v(e_0 + e_1x + ... + e_nx^n)u$. Thus, $\alpha S = v(e_0 + e_1x + ... + e_nx^n)S$ and $S\alpha = S(e_0 + e_1x + ... + e_nx^n)u$.*

For $\alpha = \sum_{i=0}^{n} a_ix^i \in S$, let

$$\alpha^\circ = (1 - a_0 - ... - a_n) + a_nx + ... + a_1x^n.$$

Note that $(\alpha^\circ)^\circ = \alpha$ for all $\alpha \in S$

Lemma 6. *Let R be a ring and let $\alpha = \sum_{i=0}^{n} e_ix^i \in S = (R, I)(x)/(x^{n+1})$ where $e_0, e_1, ..., e_n$ is a sequence of orthogonal idempotents such that $e_0 \in R$ and $e_1, ..., e_n \in I$. Then*

$$S(\alpha) = \mathbf{l}(\alpha^\circ) \ \ \textit{and} \ \ S(\alpha^\circ) = \mathbf{l}(\alpha).$$

Proof. An easy calculation shows that

$$S\alpha = Re_0 + I(e_0 + e_1)x + \cdots + I(e_0 + \cdots + e_n)x^n = \mathbf{l}(\alpha^\circ).$$

Since $(1 - e_0 - \cdots - e_n), e_1, \ldots, e_n$ is a sequence of orthogonal idempotents such that $e_1, ..., e_n \in I$ and $(1 - e_0 - \cdots - e_n) \in R$, the second equality follows. □

For the next theorem, its proof is a modification of [9, Theorem 20].

Theorem 7. *Let I be an ideal of a ring R and let $\sigma : R \to R$ is a ring endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$. R is strongly regular if and only if S is left centrally morphic for every $n \ge 0$.*

Proof. Assume that R is a unit strongly regular ring and let $\alpha \in S = (R, I)(x)/(x^{n+1})$. By Theorem 2, there exists orthogonal idempotents $e_0, e_1, ..., e_n$, where $e_1, ..., e_n \in I$ and $e_0 \in R$ such that α is equivalent to $\beta := e_0 + e_1x + ... + e_nx^n \in S$. By Lemma 6, $S\beta = l(\beta^0)$ and $S\beta^0 = l(\beta)$. Since R is strongly regular all idempotents of R are central. So β and β^0 are central in S. Then there exists $u, v \in U(S)$ such that $\alpha = u\beta v = (uv)\beta$. It follows that $S\alpha = S\beta = l(\beta^0)$ and $l(\alpha) = l(\beta) = S\beta^0$. So S is left centrally morphic.

Let $a \in R$. Since a is left morphic in R by Lemma 15, we have $Ra = l(b)$ for some $b \in R$. Let $\alpha = bx^n \in S$. Then there exists $\beta = \sum_{i=0}^{n} b_ix^i \in C(S)$ such that $l(\alpha) = S\beta$. We have $b_i \in C(I)$ for $i = 1, ..., n$ and $b_0 \in C(R)$ because $\beta \in C(S)$. By computation, one can obtain

$$l(\alpha) = l(b) + Ix + ... + Ix^n$$

and $S\beta = \{r_0b_0 + (r_0b_1 + r_1b_0)x + ... + (r_0b_n + ... + r_nb_0)x^n : \ r_i \in R \ for \ 1 \leq i \leq n \ and r_0 \in R\}$. Hence there exists $r_0 \in R, r_1 \in I$ such that $0 = r_0b_0$. Also $S\beta = l(\alpha) \subseteq l(b) + Rx + ... + Rx^n$ we have $1 = r_0b_1 + r_1b_0$. So $b_0 = b_0(r_0b_1 + r_1b_0) = b_0r_0b_1 + b_0r_1b_0 = r_0b_0b_1 + b_0r_1b_0 = b_0r_1b_0$. Therefore b_0 is regular in R. But from $l(\alpha) = S\beta$ it follows that $Rb_0 = l(b)$. Since $l(b) = Ra$ we have $Ra = Rb_0$ is an ideal of R. Thus we have proved that R is regular and every principal left ideal of R is an ideal. Hence R is strongly regular by [4, Theorem 3.2]. □

Corollary 8. *[9, Theorem 20] Let $n \geq 1$ be an integer. Then R is strongly regular if and only if $R[x]/(x^{n+1})$ is a left centrally morphic ring.*

Proof. Let $I = R$, then proof is by previous theorem. □

4. The ring $(R, I)(x)/(x^{n+1})$ is left quasi-morphic

First we fix some notation. Following Herbera [5], we define set E as following:

$$E = \{e(x) \in (R, I)[[x]] : e(x) = e + \sum_{i=1}^{\infty}(1-e)a_iex^i \ \ where \ \ e^2 = e \in R \ \ and \ \ a_i \in I \ \ for \ \ i = 1, \dots\}.$$

Fix an integer n and $(R, I)[[x]]/(x^{n+1}) \cong (R, I)(x)/(x^{n+1})$. For any $\alpha = \sum_{i=0}^{\infty} a_ix^i \in (R, I)[[x]]$, let $\overline{\alpha} = \sum_{i=0}^{n} a_ix^i$ be the image of α. We let

$$\overline{E} = \{\overline{e(x)} : e(x) \in E\}.$$

The following two lemmas are a generalization of [5, Lemma 1.3 and Lemma 1.4]. The proofs are similar to [5] but for the sake of completeness, we write them again.

Remark 9. (1) *The elements of E are idempotents of $(R, I)[[x]]$.*
(2) *Let $e(x) = e + \sum_{i=1}^{\infty}(1-e)a_iex^i \in E$, then $\mathrm{r}_{(R,I)[[x]]}(e(x)) = \mathrm{r}_R(e)(R, I)[[x]]$*

Lemma 10. *Let R be a regular ring, I be an ideal of R and $a(x) \in (R, I)[[x]]$. Then there exists power series $e(x) \in E$ and $a'(x)$ such that*

$$a(x)(R, I)[[x]] = e(x)(R, I)[[x]] + xa'(x)(R, I)[[x]] \ \ and \ \ e(x)a'(x) = 0.$$

Moreover,

$$\mathrm{l}_R(a(x)) \subseteq \mathrm{l}_R(e(x)) \cap \mathrm{l}_R(a'(x)).$$

Proof. If the zero degree term of $a(x)$ is zero then the proof is clear. Assume that

$$a(x) = a_0 + x\widetilde{a}(x)$$

with $0 \neq a_0 \in R$ and $\widetilde{a}(x) \in I(x)/(x^{n+1})$. Since R is regular there exists an element $t \in R$ such that $a_0ta_0 = a_0$. So $a_0t = e$ and $ta_0 = f$ are idempotent elements of R. Then

$$\begin{aligned} a(x)(R, I)[[x]] &= a(x)f(R, I)[[x]] + a(x)(1-f)(R, I)[[x]] \\ &= a(x)f(R, I)[[x]] + x\widetilde{a}(x)(1-f)(R, I)[[x]]. \end{aligned}$$

Moreover,

$$\begin{aligned} a(x)f(R, I)[[x]] &= (ea(x)f + (1-e)\widetilde{a}(x)fx)(R, I)[[x]] \\ &= (ea(x)f + (1-e)\widetilde{a}(x)fx)te(R, I)[[x]]. \end{aligned}$$

$ea(x)fte + 1 - e$ is a unit element of $(R, I)[[x]]$ because $ea_0fte = e$. Let $u(x)$ be inverse of $ea(x)fte + 1 - e$. Also note that $u(x) = eu(x)e + 1 - e$. Thus $ea(x)fte = (eu(x)e)^{-1}$ is a unit of $e(R, I)[[x]]e$. So we have

$$a(x)f(R, I)[[x]] = e(x)(R, I)[[x]],$$

where

$$e(x) = (ea(x)f + (1-e)a(x)f)te(eu(x)e + (1-e)) = e + \sum_{i=1}^{\infty}(1-e)b_nex^n,$$

for suitable $b_n \in I$. By definition of the set E, $e(x)$ is an element of the set E. Hence,

$$\begin{aligned} a(x)(R,I)[[x]] &= e(x)(R,I)[[x]] + x\widetilde{a}(x)(1-f)(R,I)[[x]] \\ &= e(x)(R,I)[[x]] + x(1-e(x))\widetilde{a}(x)(1-f)(R,I)[[x]]. \end{aligned}$$

If we choose $a'(x) := (1-e(x))\widetilde{a}(x)(1-f)(R,I)[[x]]$, then we are done. For the moreover part, it suffices to show that $l_R(a(x)) \subseteq l_R(e(x))$. If $r \in l_R(a(x))$ then $re = 0$, and so $r(1-e) = r$. But $e(x) = (ea(x)f + (1-e)a(x)f)te(eu(x)e + (1-e))$, so we have $re(x) = 0$. □

Lemma 11. *Let R be a regular ring, I be an ideal of R and $a(x) \in (R,I)[[x]]$. Then there exits sequence of idempotents $e_i(x) \in E$ such that, for any $n \geq 0$, there exits $a'_n(x) \in (R,I)[[x]]$ which satisfies*

$$a(x)(R,I)[[x]] = (\sum_{i=0}^{n} e_i(x)x^i)(R,I)[[x]] + a'_n(x)x^{n+1}(R,I)[[x]].$$

Moreover,
(i) $e_i(x)e_j(x) = 0$ for any $j > i \geq 0$, and
(ii) for every $0 \leq i \leq n$, $e_i(x)a'_n(x) = 0$.

Proof. We will proof it by induction on n. For the case $n = 0$, there is no need to prove because of Lemma 10. Assume $n \geq 1$ and statement is true for $n-1$. Then

$$a(x)(R,I)[[x]] = (\sum_{i=0}^{n-1} e_i(x)x^i)(R,I)[[x]] + a'_{n-1}(x)x^n(R,I)[[x]],$$

and this decomposition satisfies (i) and (ii). By applying Lemma 10 to $a'_{n-1}(x)$ we have the equality:

$$a'_{n-1}(x)x^n(R,I)[[x]] = (e_n(x)x^n)(R,I)[[x]] + a'_n(x)x^{n+1}(R,I)[[x]],$$

with $e_n(x) \in E$, $e_n(x)a'_n(x) = 0$ and $l_R(a'_{n-1}(x)) \subseteq l_R(e_n(x)) \cap l_R(a'_n(x))$. Since $e_i(x)a'_{n-1}(x) = 0$, for $0 \leq i \leq n-1$, by Remark 9, this happens iff $e_i a'_{n-1}(x) = 0$, where $e_i = e_i^2 \in R$ is the term of zero degree of $e_i(x)$. Thus $e_i(x)e_n(x) = 0$ and $e_i(x)a'_n(x) = 0$, for $0 \leq i \leq n-1$. □

Proposition 12. *Let R be a regular ring and let $S = (R,I)(x)/(x^{n+1})$ where $n \geq 0$. For $\alpha \in S$ the followings are true:*
(i) There exits a sequence of orthogonal idempotents $e_0, e_1, \ldots, e_n \in R$ and $u \in U(S)$ such that $S\alpha = S(e_0 + e_1x + \cdots + e_nx^n)u$.
(ii) There exits a sequence of orthogonal idempotents $f_0, f_1, \ldots, f_n \in R$ and $v \in U(S)$ such that $\alpha S = v(f_0 + f_1x + \cdots + f_nx^n)S$.

Proof. By symmetry it is enough to prove one of the statement. We will prove the statement (ii).
In order to apply Lemma 11, we will think S as $(R,I)[[x]]/(x^{n+1})$. Modulo the ideal (x^{n+1}) , the equality in Lemma 11 becomes

$$\alpha S = (\sum_{k=0}^{n} \overline{e_k(x)}x^k)S,$$

where $\overline{e_k(x)} \in \overline{E}$ and $\overline{e_i(x)}.\overline{e_j(x)} = 0$ whenever $j > i \geq 0$. We will proceed just as in [9, Proposition 1]. For each k with $0 \leq k \leq n$, let $\overline{e_k(x)} = e_k + \sum_{i=1}^{n}(1-e_k)a_i^{(k)}e_kx^i$. It follows that $e_ie_j = 0$ whenever $j > i \geq 0$. Now we will use the same technique in [5, Corollary 1.7]. Let $\overline{e_k(x)} = e_k + \sum_{i=1}^{n}(1-e_k)a_i^{(k)}e_kx^i$, for each k with $0 \leq k \leq n$. Then we have $e_ie_j = 0$, whenever $j > i \geq 0$. Hence $\sum_{k=0}^{n} e_kR = \bigoplus_{k=0}^{n} e_kR$. By hypothesis we can write $\bigoplus_{k=0}^{n} e_kR = eR$ where e is an idempotent element of R. Define $h_i : \bigoplus_{k=0}^{n} e_kR \to R$ by $h_i(\sum_{k=0}^{n} e_kr_k) = \sum_{k=0}^{n}(1-e_k)a_i^{(k)}e_kr_k$ which is a left multiplication by $b_i = h_i(e)$, for each i with $1 \leq i \leq n$. So $b_ie_k = h_i(e_k) = (1-e_k)a_i^{(k)}e_k$. Define v as following:

$$v := 1 + \sum_{i=1}^{n} b_ix^i.$$

Then $v \in U(S)$ and $v(\sum_{k=0}^{n} e_kx^k) = \sum_{k=0}^{n} \overline{e_k(x)}x^k$. Sequence of orthogonal idempotents $\{f_k\}_{k=0}^{n}$ can be constructed such as :

$$f_0 = e_0 \;\; and \;\; f_k = e_k(1 - f_0 - \cdots - f_{k-1}) \;\; for \;\; k = 1, \ldots, n.$$

By [5, Remark 1.6], the R-module epimorphism $g:\bigoplus_{k=0}^{n}Re_k\to\bigoplus_{k=0}^{n}Rf_k$ which is given by $g(\sum_{k=0}^{n}r_ke_k)=\sum_{k=0}^{n}r_kf_k$is an isomorphism. Write $\bigoplus_{k=0}^{n}Re_k=Ra$ and $\bigoplus_{k=0}^{n}Rf_k=Rb$, where a and b are idempotents of R. Say $c=g(a)$ and $d=g^{-1}(b)$. Then g and g^{-1} are the right multiplication by c, and d, respectively. Hence

$$\sum_{k=0}^{n}e_kx^k=\sum_{k=0}^{n}g^{-1}(f_k)x^k=\sum_{k=0}^{n}f_kdx^k=(\sum_{k=0}^{n}f_kx^k)d$$

and

$$\sum_{k=0}^{n}f_kx^k=\sum_{k=0}^{n}g(e_k)x^k=\sum_{k=0}^{n}e_kcx^k=(\sum_{k=0}^{n}e_kx^k)c.$$

So $(\sum_{k=0}^{n}e_kx^k)S=(\sum_{k=0}^{n}f_kx^k)S$. Hence $\alpha S=v(\sum_{k=0}^{n}e_kx^k)S=v(\sum_{k=0}^{n}f_kx^k)S$. □

Theorem 13. *Let R be a regular ring and let $n\geq 0$. Then $S=(R,I)(x)/(x^{n+1})$ is a quasi-morphic ring.*

Proof. By symmetry, we only show that S is a left quasi morphic ring. Let $\alpha\in S$. By Proposition 12,

$$S\alpha=S(e_0+e_1x+...+e_nx^n)u$$

and

$$\alpha S=v(f_0+f_1+...+f_nx^n)S,$$

where u,v are unit elements of S and $\{e_i\}_{i=0}^{n}$, $\{f_i\}_{i=0}^{n}$ are sequences of orthogonal idempotents of R. Let $\beta=\sum_{i=0}^{n}e_ix^i$ and $\gamma=\sum_{i=0}^{n}f_ix^i$. Then, by [9, Lemma 3],

$$S\alpha=(S\beta)u=l(\beta^0)u=l(u^{-1}\beta^0),$$

$$l(\alpha)=l(v\gamma)=l(\gamma)v^{-1}=(S\gamma^0)v^{-1}=S(\gamma^0v^{-1}).$$

So α is a left quasi-morphic in S. □

Corollary 14. *If R is regular and $n\geq 0$, then the matrix rings over $(R,I)(x)/(x^{n+1})$ are all quasi-morphic.*

Proof. If R is regular then $M_k(R)$ is regular for each $k\geq 1$. So $M_k((R,I)(x)/(x^{n+1}))\cong M_k(R,I)(x)/(x^{n+1})$ is quasi-morphic by Theorem 13. □

The following theorem generalizes [9, Lemma 10].

Lemma 15. *Let $n\geq 0$ be an integer. If $S=(R,I)(x)/(x^{n+1})$ is left quasi-morphic (resp., left morphic), then so is R.*

Proof. Let $a\in R$ and let $\alpha=a\in S$. Since α is left quasi-morphic in S, $S\alpha=\mathbf{l}(\beta)$ and $\mathbf{l}(\alpha)=S\gamma$, where $\beta=\sum_{i=0}^{n}b_ix^i$ and $\gamma=\sum_{i=0}^{n}c_ix^iinS$. But

$$\mathbf{l}(\alpha)=\mathbf{l}(a)+\mathbf{l}(a)x+\cdots+\mathbf{l}(a)x^n \quad and$$

$$S\gamma=\{r_0c_0+(r_0c_1+r_1c_0)x+\cdots+(r_0c_n+r_1c_{n-1}+\cdots+r_nc_0)x^n:\ \ r_0\in R\ \ r_1,\ldots,r_n\in I\}.$$

So it follows from $\mathbf{l}(\alpha)=S\gamma$ that $\mathbf{l}(a)=Rc_0$. On the other hand, $\alpha\beta=0$ clearly implies that $Ra\subseteq\mathbf{l}b_0$. Moreover,

$$(\mathbf{l}(b_0)\cap\cdots\cap\mathbf{l}(b_n))+(\mathbf{l}(b_0)\cap\cdots\cap\mathbf{l}(b_{n-1}))x+\cdots+\mathbf{l}(b_0)x^n$$

$\mathbf{l}(\beta)=S\alpha=Ra+Iax+\cdots+Iax^n$.
So $\mathbf{l}(b_0)\subseteq Ra$. Hence $Ra=\mathbf{l}(b_0)$. So a is left quasi-morphic in R. If α is left morphic in S, then β and γ can be chosen to be the same. Thus, a is left morphic in R since $b_0=c_0$ in this case. □

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