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A Note on An Almost Contanct Metric Manifold with A Type of Semi-symmetric Non-metric Connection

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Abstract
In this paper, we focus on an almost contact metric manifold admitting a type of semi-symmetric non-metric connection. We find the expression for the curvature tensor of such a manifold. Furhermore, we study the properties of the curvature tensor and the projective curvature tensor.

Keywords: *Almost contact metric manifold; Semi-symmetric non-metric connection; Curvature tensor.*

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1. Introduction

An odd dimensional differentiable manifold M of class C^∞ is said to have an almost contact structure by Gray *[9]* if the structural group of its tangent bundle reduces to $U(n) \times 1$. In 1960, Sasaki *[15]* showed that the notions of an almost contact structure and a (ϕ, ξ, η)- structure satisfying certain conditions are equivalent. There are many different types of almost contact structures defined in the literature such as cosymplectic, Sasakian, almost cosymplectic, quasi Sasakian, normal, $\alpha-$Kenmotsu, $\alpha-$Sasakian, trans-Sasakian, etc... [2, 3, 11, 13].

Let (M^n, g) be an $n(= 2m + 1)-$dimensional Riemannian manifold of class C^∞. Let there exist in M^n a 1-form η, a vector field ξ and a vector valued linear function ϕ such that

$$\phi^2 = -X + \eta(X)\xi, \tag{1.1}$$

for any vector field X. Then M^n is called *an almost contact manifold* and the system (ϕ, ξ, η) is called *an almost contact structure to M^n*.

From (1.1), it follows [3]

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1. \tag{1.2}$$

If the Riemannian metric g in M^n satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

for any vector fields X and Y in M^n, then (M^n, g) is called *an almost contact metric manifold* and g is called *a compatible metric [15].* In view of (1.2) and (1.3), we get

$$g(\xi, Y) = \eta(Y) \tag{1.4}$$

and

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$$g(X, \phi Y) = g(\phi X, Y). \tag{1.5}$$

The notion of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [8]. Let $\overline{\nabla}$ be a linear connection in an n-dimensional Riemannian manifold with Riemannian metric g. A linear connection $\overline{\nabla}$ on a Riemannian manifold M^n is called *a semi-symmetric connection* if the torsion tensor T of the connection $\overline{\nabla}$

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \tag{1.6}$$

satisfies

$$T(X, Y) = \pi(Y)X - \pi(X)Y, \tag{1.7}$$

where π is a 1-form. The connection $\overline{\nabla}$ is a metric connection if there is a Riemannian metric g in M^n such that $\overline{\nabla} g = 0$, otherwise it is non-metric. In 1932, Hayden [10] defined a semi- symmetric metric connection on a Riemannaian manifold and Yano [17] developed it. Several authors such as Pravonovich [14], Agashe and Chafle **[1]**, Liang [12] and Sengupta, De and Binh [16] introduced a semi-symmetric non-metric connection in different ways and this connection is studied by many authors [7, 16, 18], etc.... Recently, Chaubey and Ojha [4, 5] defined a new type of semi-symmetric non-metric connection in an almost contact metric manifold. In 2011, These authors [6] studied some properties of a semi-symmetric non-metric connection in a Kenmotsu manifold.

In the present paper we consider a semi-symmetric non-metric connection in the sense of Agashe and Chafle **[1]** and define a semi-symmetric non-metric connection in an almost contact metric manifold by identifying the 1-form π of (1.7) with a $1-$ form η, i.e., by setting

$$T(X, Y) = \eta(Y)X - \eta(X)Y \tag{1.8}$$

and deal with almost contact manifolds admitting a type of semi symmetric non-metric connection $\overline{\nabla}$ satisfying the condition (1.8) and

$$\left(\overline{\nabla}_X T\right)((Y, Z) = \delta(X)T(Y, Z) + w(X)g(Y, Z)\rho, \tag{1.9}$$

where δ and w are non-zero 1-forms defined by

$$\delta(X) = g(X, P), \;\; w(X) = g(X, \rho) \tag{1.10}$$

and the 1-form w satisfies

$$w(R(X, Y)Z) = 0. \tag{1.11}$$

In Section 3 we find the expression for curvature tensor of $\overline{\nabla}$ and deduce some properties of the curvature tensor. It is proved that if the curvature tensor of $\overline{\nabla}$ vanishes, the $1-$ form η of the manifold is closed, the vector field ξ is irrotational and the integral curves of ξ are geodesics. In Section 4 we deal with the projective curvature tensor of $\overline{\nabla}$ and we obtain that if the Ricci tensor of $\overline{\nabla}$ is symmetric, the projective curvature tensors of the manifold with respect to the Levi-Civita connection and the semi symmetric connection are equal. Next we prove that if the Ricci tensor of $\overline{\nabla}$ vanishes, then the projective curvature tensor of the manifold is equal to the curvature tensor of the manifold with respect to the semi symmetric non-metric connection. Finally, if the curvature tensor of $\overline{\nabla}$ vanishes, we show that it is projectively flat, its scalar curvature tensor r is not zero and the vector field ξ is a Ricci principal direction with corresponding eigen value r and we also obtain the expressions of the curvature tensor R and the Ricci tensor S of this manifold.

2. Preliminaries

The relation between the semi-symmetric non-metric connection $\overline{\nabla}$ and the Levi-Civita connection ∇ of (M^n, g) is defined in the following form by Agashe and Chafle [1]

$$\overline{\nabla}_X Y = \nabla_X Y + \pi(Y)X \tag{2.1}$$

where π is a 1-form.

Further, a relation between the curvature tensors R and $\overline{R}$ of type (1,3) of the connections ∇ and $\overline{\nabla}$ respectively are given by [1]

$$\overline{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X \tag{2.2}$$

where α is a tensor field of type (0,2) defined by

$$\alpha(X,Y) = (\nabla_X \pi)(Y) - \pi(X)\pi(Y) = \left(\overline{\nabla}_X \pi\right)(Y). \tag{2.3}$$

Contracting (2.2) we have

$$\overline{S}(Y,Z) = S(Y,Z) + (1-n)\alpha(Y,Z) \tag{2.4}$$

where $\overline{S}$ and S denote the Ricci tensors of the semi-symmetric non-metric connection and Levi-Civita connection, respectively.

3. Almost contact metric manifolds admitting a special type of semi-symmetric non-metric connection

In this section we consider an almost contact metric manifold admitting semi-symmetric non-metric connection whose torsion tensor T is given by (1.8) and it satisfies (1.9). Then, from (1.8), contracting over X, we get

$$\left(C_1^1 T\right)(Y) = (n-1)\eta(Y). \tag{3.1}$$

From (3.1), it follows that

$$\left(\nabla_X C_1^1 T\right)(Y) = (n-1)\left(\nabla_X \eta\right)(Y). \tag{3.2}$$

Contracting (1.9) and using (3.1), we get

$$\left(\overline{\nabla}_X C_1^1 T\right)(Y) = (n-1)\delta(X)\eta(Y) + w(X)w(Y). \tag{3.3}$$

Using (3.2), from (3.3), we obtain

$$\left(\overline{\nabla}_X \eta\right)(Y) = \delta(X)\eta(Y) + \frac{1}{n-1}w(X)w(Y). \tag{3.4}$$

Since $\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X$, from $\left(\overline{\nabla}_X \eta\right)(Y) = \overline{\nabla}_X \eta(Y) - \eta\left(\overline{\nabla}_X Y\right)$, it follows that

$$\left(\overline{\nabla}_X \eta\right)(Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y). \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$(\nabla_X \eta)(Y) = \eta(X)\eta(Y) + \delta(X)\eta(Y) + \frac{1}{n-1}w(X)w(Y), \tag{3.6}$$

in virtue of (2.3) and (3.5), we get

$$\alpha(X,Y) = \delta(X)\eta(Y) + \frac{1}{n-1}w(X)w(Y). \tag{3.7}$$

Now, using (2.2) and (3.7) , the expression of the curvature tensor $\overline{R}$ with respect to the connection $\overline{\nabla}$ can be written as

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ \left\{\delta(X)\eta(Z) + \frac{1}{n-1}w(X)w(Z)\right\}Y \\ &- \left\{\delta(Y)\eta(Z) + \frac{1}{n-1}w(Y)w(Z)\right\}X.\end{aligned} \tag{3.8}$$

Thus we can state the following thorem:

Theorem 3.1. *The curvature tensor with respect to $\overline{\nabla}$ of an almost contact metric manifold admitting the semi-symmetric non-metric connection whose torsion tensor T is given by (1.8) satisfies (1.9) is of the form (3.8).*

Replacing X with Y in (3.8), from (3.8), it is clear that

$$\overline{R}(X,Y)Z = -\overline{R}(Y,X)Z. \tag{3.9}$$

From (3.8), we get

$$\begin{aligned}\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = &\{\delta(Z)\eta(Y) - \delta(Y)\eta(Z)\}X \\ &+ \{\delta(X)\eta(Z) - \delta(Z)\eta(X)\}Y \\ &+ \{\delta(Y)\eta(X) - \delta(X)\eta(Y)\}Z.\end{aligned} \tag{3.10}$$

This is the first Bianchi identity with respect to $\overline{\nabla}$.
Again from (3.8), we get by contracting X

$$\overline{S}(Y,Z) = S(Y,Z) - (n-1)\delta(Y)\eta(Z) - w(Y)w(Z). \tag{3.11}$$

In (3.11) we put $Y = Z = e_i$, $1 \leq i \leq n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold. Then summing over i we get

$$\overline{r} = r - (n-1)\delta(\xi) - w(\rho) \tag{3.12}$$

where $\overline{r}$ and r denote the scalar curvatures of this manifold with respect to $\overline{\nabla}$ and ∇, respectively.
From (3.11), it follows that $\overline{S}$ is symmetric if and only if

$$\delta(Y)\eta(Z) = \delta(Z)\eta(Y). \tag{3.13}$$

Putting $Z = \xi$ in (3.13) and using (1.2), we get

$$\delta(Y) = a\eta(Y) \tag{3.14}$$

where $a = \delta(\xi)$.
Now let us assume that $\overline{S}$ is symmetric. Then (3.10) and (3.14) we get

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0. \tag{3.15}$$

Conversely, we assume that (3.15) holds, then in virtue of (3.10) we have

$$\{\delta(Z)\eta(Y) - \delta(Y)\eta(Z)\}X + \{\delta(X)\eta(Z) - \delta(Z)\eta(X)\}Y + \{\delta(Y)\eta(X) - \delta(X)\eta(Y)\}Z = 0. \tag{3.16}$$

Contracting X, from (3.16), it follows that

$$\delta(Y)\eta(Z) - \delta(Z)\eta(Y) = 0. \tag{3.17}$$

Hence by (3.13), $\overline{S}$ is symmetric.
Thus, we can state the following theorem:

Theorem 3.2. *A necessary and sufficient condition for the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric non-metric connection whose torsion tensor is given by (1.8) satisfies (1.9) to be symmetric is*

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0.$$

From (3.8) and (3.14), it follows that

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ \left\{a\eta(X)\eta(Z) + \frac{1}{n-1}w(X)w(Z)\right\}Y \\ &- \left\{a\eta(Y)\eta(Z) + \frac{1}{n-1}w(Y)w(Z)\right\}X.\end{aligned} \tag{3.18}$$

We now define a covariant curvature tensor $\overline{R}$ of type (0,4) by

$$\overline{R}(X,Y,Z,V) = g(\overline{R}(X,Y)Z,V). \tag{3.19}$$

In virtue of (3.19), from (3.18), we have

$$\begin{aligned}\overline{R}(X,Y,Z,V) = R(X,Y,Z,V) + \left\{a\eta(X)\eta(Z) + \frac{1}{n-1}w(X)w(Z)\right\} g(Y,V)\\ - \left\{a\eta(Y)\eta(Z) + \frac{1}{n-1}w(Y)w(Z)\right\} g(X,V).\end{aligned} \tag{3.20}$$

If $\overline{R} = 0$, then we get

$$\begin{aligned}R(X,Y,Z,V) = \left\{a\eta(Y)\eta(Z) + \frac{1}{n-1}w(Y)w(Z)\right\} g(X,V)\\ - \left\{a\eta(X)\eta(Z) + \frac{1}{n-1}w(X)w(Z)\right\} g(Y,V)\end{aligned} \tag{3.21}$$

and

$$S(Y,Z) = (n-1)a\eta(Z)\eta(Y) + w(Z)w(Y). \tag{3.22}$$

Thus, from (3.21) and (3.22), it follows that

$$R(X,Y,Z,V) = \frac{1}{(n-1)}\left\{S(Y,Z)g(X,V) - S(X,Z)g(Y,V)\right\}. \tag{3.23}$$

Putting $V = \rho$ in (3.23), we obtain

$$w(R(X,Y)Z) = \frac{1}{(n-1)}\left\{S(Y,Z)w(X) - S(X,Z)w(Y)\right\}. \tag{3.24}$$

Using (1.11) and (3.22), we get

$$a\left\{\eta(Y)w(X) - \eta(X)w(Y)\right\} = 0. \tag{3.25}$$

Since δ is a non-zero 1-form, from the above equation, it follows that

$$\eta(Y)w(X) = \eta(X)w(Y). \tag{3.26}$$

Hence by (1.2) and (3.26), we have

$$w(Y) = b\eta(Y) \tag{3.27}$$

where $b = w(\xi)$.

From (3.6), (3.14) and (3.27) , it follows that

$$(\nabla_X\eta)(Y) = \left[1 + a + \frac{b^2}{n-1}\right]\eta(X)\eta(Y). \tag{3.28}$$

Therefore, we get

$$(\nabla_X\eta)(Y) = (\nabla_Y\eta)(X). \tag{3.29}$$

Hence by (3.29), we find that the 1-form η is closed. From (3.14) and (3.27), it follows that the 1-forms δ and w are also closed.

Furthermore, using (3.29),we have

$$g(Y,\nabla_X\xi) = g(X,\nabla_Y\xi), \tag{3.30}$$

which implies that the vector field ξ is irrotational. Since $g(\xi, \xi) = 1$, then

$$g(X, \nabla_\xi \xi) = 0, \tag{3.31}$$

that is, $\nabla_\xi \xi = 0$ which implies that the integral curves of ξ are geodesics.

Hence we can state:

Theorem 3.3. *If an almost contact metric manifold admits a semi-symmetric non-metric connection whose torsion tensor T is given by (1.8) satisfies (1.9) and whose curvature tensor vanishes, then the $1-$ form η of the manifold is closed, the vector field ξ is irrotational and the integral curves of ξ are geodesics.*

4. Projective curvature tensor

The projective curvature tensor of type $(1, 3)$ of an almost contact metric manifold with respect to the semi-symmetric non-metric connection is defined by

$$\overline{P}(X, Y)Z = \overline{R}(X, Y)Z - \frac{1}{n-1}\left\{\overline{S}(Y, Z)X - \overline{S}(X, Z)Y\right\}. \tag{4.1}$$

If, in particular, $\overline{S}$ is symmetric, then we already have $\delta(Y) = \delta(\xi)\eta(Y)$. Thus, using (3.8) and (3.11), we get from (4.1)

$$\overline{P}(X, Y)Z = P(X, Y)Z, \tag{4.2}$$

where $P(X, Y)Z$ is the projective curvature tensor of the manifold with respect to the Levi-Civita connection defined as

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}\left\{S(Y, Z)X - S(X, Z)Y\right\}. \tag{4.3}$$

So, we have

Theorem 4.1. *If the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric non-metric connection whose torsion tensor T is given by (1.8) satisfies (1.9) is symmetric, then the projective curvature tensors of the manifold with respect to the Levi-Civita connection and the semi symmetric non-metric connection are equal.*

In virtue of (4.2) and (4.3) we can state the following:

Corollary 4.1. *If the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric non-metric connection whose torsion tensor T is given by (1.8) satisfies (1.9) is symmetric, then it is satisfied the conditions:*

i $\overline{P}(X, Y)Z + \overline{P}(Y, Z)X = 0,$

ii $\overline{P}(X, Y)Z + \overline{P}(Y, Z)X + \overline{P}(Z, X)Y = 0.$

Next, if in particular $\overline{S} = 0$, then from (4.1) and (4.2) we have

$$P(X, Y)Z = \overline{R}(X, Y)Z. \tag{4.4}$$

So we can state:

Theorem 4.2. *If the Ricci tensor of an almost contact metric manifold with respect to the semi-symmetric non-metric connection whose torsion tensor T is given by (1.8) satisfies (1.9) vanishes, then the projective curvature tensor of the manifold is equal to the curvature tensor of the manifold with respect to the semi symmetric non-metric connection.*

If $\overline{R} = 0$, then from (3.18) and (3.27), it follows that

$$R(X, Y)Z = \left[a - \frac{b^2}{n-1}\right]\eta(Z)\left\{\eta(X)Y - \eta(Y)X\right\}. \tag{4.5}$$

Contracting (4.5) we get

$$S(Y, Z) = \left[b^2 - a(n-1)\right]\eta(Y)\eta(Z). \tag{4.6}$$

Further contraction yields

$$r = b^2 - a(n-1). \tag{4.7}$$

Let L be the symmetric linear operator such that

$$S(Y,Z) = g(LY,Z). \tag{4.8}$$

Thus from (4.6) and (4.8), we get

$$g(LY,Z) = r\eta(Y)\eta(Z). \tag{4.9}$$

Putting $Y = \xi$ in (4.9) and using (1.2), we obtain

$$L\xi = r\xi. \tag{4.10}$$

Hence, using (4.3) we have the following theorem:

Theorem 4.3. *If an almost contact metric manifold admits a semi-symmetric non-metric connection $\overline{\nabla}$ with the vector field ξ whose curvature tensor $\overline{R}$ vanishes and whose torsion tensor T is given by (1.8) satisfies (1.9), then*

i *The curvature tensor R and the Ricci tensor S of this manifold are respectively given by (4.5) and (4.6). Its scalar curvature is also not zero.*

ii *It is projectively flat.*

iii *The vector field ξ is a Ricci principal direction with corresponding eigen value r.*

Moreover, from (1.2), (1.4), (4.5) and (4.6) we can state the following theorem:

Theorem 4.4. *If an almost contact metric manifold admits a semi-symmetric non-metric connection whose torsion tensor T is given by (1.8) satisfies (1.9) and whose curvature tensor vanishes, then we have*

i $\eta(R(X,Y)Z) = 0$,

ii $R(X,Y)\xi = \left[a - \frac{b^2}{n-1}\right]\{\eta(X)Y - \eta(Y)X\}$,

iii $S(Y,\xi) = r\eta(Y)$.

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