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Tychonoff Objects in the Topological Category of Cauchy Spaces

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Abstract
There are various forms of Tychonoff objects for an arbitrary set-based topological category. In this paper, any explicit characterization of each of the Tychonoff Objects is given in the topological category of Cauchy spaces. Moreover, we characterize each of them for the category of Cauchy spaces and investigate the relationships among the various T_i, $i = 0, 1, 2, 3, 4$, $PreT_2$, and $\mathbf{T}_2$ (we will refer to it as the usual one) structures are examined in this category.

Keywords: *Topological category; Cauchy space; Cauchy map; Separation.*

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1. Introduction

In general topology and analysis, a Cauchy space is a generalization of metric spaces and uniform spaces. The theory of Cauchy spaces was initiated by H. J. Kowalsky [20]. Cauchy spaces were introduced by H. Keller [17] in 1968.

In 1970, the study of regular Cauchy completions was initiated by J. Ramaley and O. Wyler [31]. Later, D. C. Kent and G. D. Richardson ([18, 19]) characterized the T_3 Cauchy spaces which have T_3 completions and constructed a regular completion functor.

In 1968, Keller [17] introduced the axiomatic definition of Cauchy spaces, which is given briefly in the preliminaries section.

Filter spaces are generalizations of Cauchy spaces. If we exclude the last of three Keller's [17] axioms for a Cauchy space, then the resulting space is what we call a filter space. In [13], it is shown that the category FIL of filter spaces is isomorphic to the category of filter meretopic spaces which were introduced by Katětov [16]. The category of Cauchy spaces is also known to be a bireflective, finally dense subcategory of FIL [30].

All our preliminary information on Cauchy spaces and more information can be found in [24].

The notions of "closedness" and "strong closedness" in set based topological categories are introduced by Baran [2, 4] and it is shown in [9] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [14] in some well-known topological categories. Moreover, various generalizations of each of T_i, $i = 0, 1, 2$ separation properties for an arbitrary topological category over $\boldsymbol{Set}$, the category of sets are given and the relationship among various forms of each of these notions are investigated by Baran in [2, 7, 8, 10, 11].

The main goal of this paper is

1. to give the characterization of each of the Tychonoff objects in the topological category of Cauchy spaces,

2. to examine how these generalizations are related, and

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3. to show that specific relationships that arise among the various T_i, $i = 0, 1, 2, 3, 4$, $PreT_2$, and $\mathbf{T}_2$ (we will refer to it as the usual one) structures are examined in the topological category of Cauchy spaces.

2. Preliminaries

The followings are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful, amnestic and transportable), has small (i.e., sets) fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure [1, 5, 10, 26, 29].

Recall in [1, 29], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in$Ob $\mathcal{E}$), a topological category, is discrete iff every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

A filter on a set X is a collection of subsets of X, containing X, which is closed under finite intersection and formation of supersets (it may contain $\emptyset$). Let $\mathbf{F}(X)$ denote the set of filters on X. If $\alpha,\ \beta\ \in \mathbf{F}(X)$, then $\beta \geq \alpha$ if and only if for each U$\in \alpha$, $\exists$V$\in \beta$ such that $V \subseteq U$, that is equivalent to $\beta \supset \alpha$. This defines a partial order relation on $\mathbf{F}(X)$. $\dot{x} = [\{x\}]$ is the filter generated by the singleton set $\{x\}$ where $[\cdot]$ means generated filter and $\alpha \cap \beta = [\{\ \mathrm{U} \cup \mathrm{V} \mid \mathrm{U} \in \alpha, \mathrm{V} \in \beta\ \}]$. If U$\cap$V$\neq \emptyset$, for all U$\in \alpha$ and V$\in \beta$, then $\alpha \vee \beta$ is the filter $[\{\mathrm{U}\ \cap \mathrm{V} \mid \mathrm{U} \in \alpha, \mathrm{V} \in \beta\ \}]$. If $\exists$U$\in \alpha$ and V$\in \beta$ such that U$\cap$V=$\emptyset$, then we say that $\alpha \vee \beta$ fails to exist.

Let A be a set and q be a function on A that assigns to each point x of A a set of filters (proper or not, where a filter δ is proper iff δ does not contain the empty set, $\emptyset$, i.e., $\delta \neq [\emptyset]$) (the filters converging to x) is called a *convergence structure on* A ((A, q) a *convergence space* (in [29], it is called a convergence space)) iff it satisfies the following three conditions ([28] p. 1374 or [29] p. 142):

1. $[x] = [\{x\}] \in q(x)$ for each $x \in A$ (where $[F] = \{B \subset A\ :\ F \subset B\}$).
2. $\beta \supset \alpha \in q(x)$ implies $\beta \in q(x)$ for any filter β on A.
3. $\alpha \in q(x) \Rightarrow \alpha \cap [x] \in q(x)$.

A map $f : (A, q) \to (B, s)$ between two convergence spaces is called *continuous* iff $\alpha \in q(x)$ implies $f(\alpha) \in s(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D)\ : D \in \alpha\}$). The category of convergence spaces and continuous maps is denoted by **Con** (in [29] **Conv**).

For filters α and β we denote by $\alpha \cup \beta$ the smallest filter containing both α and β.

Definition 2.1. (cf. [17]) Let A be a set and $K \subset \mathbf{F}(A)$ be subject to the following axioms:

1. $[x] = [\{x\}] \in K$ for each $x \in A$ (where $[x] = \{B \subset A\ :\ x \in B\}$);
2. $\alpha \in K$ and $\beta \geq \alpha$ implies $\beta \in K$ (i.e., $\beta \supset \alpha \in K$ implies $\beta \in K$ for any filter β on A);
3. if $\alpha,\ \beta \in K$ and $\alpha \vee \beta$ exists (i.e., $\alpha \cup \beta$ is proper), then $\alpha \cap \beta \in K$.

Then K is a precauchy (Cauchy) structure if it obeys 1-2 (resp. 1-3) and the pair (A, K) is called a precauchy space (Cauchy space), resp. Members of K are called Cauchy filters. A map $f : (A, K) \to (B, L)$ between Cauchy spaces is said to be Cauchy continuous (Cauchy map) iff $\alpha \in K$ implies $f(\alpha) \in L$ (where $f(\alpha)$ denotes the filter generated by $\{f(D)\ : D \in \alpha\}$). The concrete category whose objects are the precauchy (Cauchy) spaces and whose morphisms are the Cauchy continuous maps is denoted by **PCHY** (**CHY**), respectively.

Definition 2.2. A source $\{f_i : (A, K) \to (A_i, K_i)\ ,\ i \in I\}$ in **CHY** is an initial lift iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$ [24, 30, 32].

Definition 2.3. An epimorphism $f : (A, K) \to (B, L)$ in **CHY** (equivalently, f is surjective) is a final lift iff $\alpha \in L$ implies that there exists a finite sequence $\alpha_1, ..., \ \alpha_n$ of Cauchy filters in K such that every member of α_i intersects every member of α_{i+1} for all $i < n$ and such that $\bigcap_{i=1}^{n} f(\alpha_i) \subset \alpha$ [24, 30, 32].

Definition 2.4. Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p ([2] p. 334), i.e., two disjoint copies of B identified at p, i.e., the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**). An epi sink $\{i_1,\ i_2 : (B, K) \to (B \vee_p B, L)\ \}$, where $i_1,\ i_2$ are the canonical injections, in **CHY** is a final lift if and only if the following statement holds. For any filter α on the wedge $B \vee_p B$, where either $\alpha \supset i_k(\alpha_1)$ for some $k = 1, 2$ and some $\alpha_1 \in K$, or $\alpha \in L$, we have that there exist Cauchy filters $\alpha_1, \alpha_2 \in K$ such that every member of α_1 intersects every member of α_2 (i.e., $\alpha_1 \cup \alpha_2$ is proper) and $\alpha \supset i_1\alpha_1 \cap i_2\alpha_2$. This is a special case of Definition 2.3.

Definition 2.5. The discrete structure (A, K) on A in $\mathbf{CHY}$ is given by $K = \{[a] \mid a \in A\} \cup \{[\emptyset]\}$ [24, 30].

Definition 2.6. The indiscrete structure (A, K) on A in $\mathbf{CHY}$ is given by $K = F(A)$ [24, 30].

$\mathbf{CHY}$ is a normalized topological category. The category of Cauchy spaces is cartesian closed, and contains the category of uniform spaces as a full subcategory [30].

3. T_2-Objects

Recall, in [2, 11], that there are various ways of generalizing the usual T_2 separation axiom to topological categories. Moreover, the relationships among various forms of T_2-objects are established in [11].

Let B be a nonempty set, $B^2 = B \times B$ be cartesian product of B with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of B^2 identified along the diagonal. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1$(or $(x, y)_2$) if (x, y) is in the first (or second) component of $B^2 \vee_\Delta B^2$, respectively. Clearly, $(x, y)_1 = (x, y)_2$ iff $x = y$ [2].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$. Note that $\pi_1 S = \pi_{11} = \pi_1 A$, $\pi_2 S = \pi_{21} = \pi_2 A$, $\pi_3 A = \pi_{12}$, and $\pi_3 S = \pi_{22}$, where $\pi_k : B^3 \to B$ the k-th projection $k = 1,\ 2,\ 3$ and $\pi_{ij} = \pi_i + \pi_j : B^2 \vee_\Delta B^2 \to B$, for $i, j \in \{1, 2\}$ [2].

Definition 3.1. (cf. [2, 4, 10, 11]) Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.

1. X is $\overline{T_0}$ iff the initial lift of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.

2. X is T_0' iff the initial lift of the $\mathcal{U}$-source $\{id : B^2 \vee_\Delta B^2 \to \mathcal{U}(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ and $\mathcal{D}(B^2)$ is the discrete structure on B^2. Here, i_1 and i_2 are the canonical injections.

3. X is T_0 iff X does not contain an indiscrete subspace with (at least) two points [25, 34].

4. X is T_1 iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete.

5. X is $Pre\overline{T_2}$ iff the initial lifts of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ coincide.

6. X is $PreT_2'$ iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ coincide, where i_1 and i_2 are the canonical injections.

7. X is $\overline{T_2}$ iff X is $\overline{T_0}$ and $Pre\overline{T_2}$.

8. X is T_2' iff X is T_0' and $PreT_2'$.

9. X is ST_2 iff Δ, the diagonal, is strongly closed in X^2.

10. X is ΔT_2 iff Δ, the diagonal, is closed in X^2.

11. X is KT_2 iff X is T_0' and $Pre\overline{T_2}$.

12. X is LT_2 iff X is $\overline{T_0}$ and $PreT_2'$.

13. X is MT_2 iff X is T_0 and $PreT_2'$.

14. X is NT_2 iff X is T_0 and $Pre\overline{T_2}$.

Remark 3.1. Note that for the category $\boldsymbol{Top}$ of topological spaces, $\overline{T_0}$, T_0', T_0, or T_1, or $Pre\overline{T_2}$, $PreT_2'$, or all of the T_2's in Definition 3.1 reduce to the usual T_0, or T_1, or $PreT_2$ (where a topological space is called $PreT_2$ if for any two distinct points, if there is a neighbourhood of one missing the other, then the two points have disjoint neighbourhoods), or T_2 separation axioms, respectively [2].

Definition 3.2. A Cauchy space (A, K) is said to be $\mathbf{T}_2$ (we will refer to it as the usual one) if and only if $x = y$, whenever $[x] \cap [y] \in K$ [33].

Theorem 3.1. *(cf. [21]) Let (A,K) be a Cauchy space. Then,*
(1)(A,K) in $\mathbf{CHY}$ is $\overline{T}_0$ iff it is T_0 iff it is T_1 iff for each distinct pair x and y in A, we have $[x]\cap[y]\notin K$.
(2) All objects (A,K) in $\mathbf{CHY}$ are T_0'.
(3) All objects (A,K) in $\mathbf{CHY}$ are $Pre\overline{T}_2$.
(4) (A,K) is $PreT_2'$ iff for each pair of distinct points x and y in A, we have $[x]\cap[y]\in K$(equivalently, for each finite subset F of A, we have $[F]\in K$).
(5) (A,K) is $\overline{T}_2$ iff for each distinct pair x and y in A, we have $[x]\cap[y]\notin K$.
(6) (A,K) is T_2' iff for each distinct points x and y in A, we have $[x]\cap[y]\in K$(equivalently, for each finite subset F of A, we have $[F]\in K$).

Remark 3.2. (cf. [21])

(1) If a Cauchy space (A,K) is $\overline{T}_0$ or $T_0\,(T_1)$ then it is T_0'. However, the converse is not true generally. For example, let $A=\{x,y\}$ and $K=\{[x],\ [y],\ [\{x,\ y\}],\ [\emptyset]\}$. Then (A,K) is T_0' but it is not $\overline{T}_0$ or $T_0\,(T_1)$.

(2) If a Cauchy space (A,K) is $PreT_2'$ then it is $Pre\overline{T}_2$. However, the converse is not true, in general. For example, let $A=\{x,y\}$ and $K=\{[x],\ [y],\ [\emptyset]\}$. Then (A,K) is $Pre\overline{T}_2$ but it is not $PreT_2'$.

Remark 3.3. (A,K) be in $\mathbf{CHY}$. By Theorem 3.1, the following are equivalent:

(a) (A,K) is $\overline{T}_2$ and T_2'.
(b) A is a point or the empty set [21].

Corollary 3.1. *Let (A,K) be in $\mathbf{CHY}$. (A,K) is ST_2 iff it is ΔT_2 iff for each pair of distinct points x and y in A and for any $\alpha,\ \beta\in K$, $\alpha\cup\beta$ is improper if $\alpha\subset[x]$ and $\beta\subset[y]$ [21].*

Remark 3.4. (A,K) be in $\mathbf{CHY}$. By Remark 4.5 (2) of [22], (A,K) is $\overline{T}_2$ iff (A,K) is ST_2 or ΔT_2.

Remark 3.5. ([3], p. 106) Let α and β be filters on A. If $f: A\to B$ is a function, then $f(\alpha\cap\beta)=f\alpha\cap f\beta$.

Let (A,K) be in $\mathbf{CHY}$, and F be a nonempty subset of A. Let $q:(A,K)\to(A/F,L)$ be the quotient map that identifying F to a point, $*$ [2].

Theorem 3.2. *(cf. [23])*
(1) If (A,K) is T_2', then $(A/F,L)$ is T_2'.
(2) If (A,K) is $\overline{T}_2$, then $(A/F,L)$ is $\overline{T}_2$.
(3) If (A,K) is $Pre\overline{T}_2$, then $(A/F,L)$ is $Pre\overline{T}_2$.
(4) If (A,K) is $PreT_2'$, then $(A/F,L)$ is $PreT_2'$

Theorem 3.3. *Let (A,K) be in $\mathbf{CHY}$. $\emptyset\neq F\subset A$ is closed iff for each $a\in A$ with $a\notin F$ and for all $\alpha\in K$, $\alpha\cup[F]$ is improper or $\alpha\nsubseteq[a]$ [21].*

Theorem 3.4. *Let (A,K) be in $\mathbf{CHY}$. $\emptyset\neq F\subset A$ is strongly closed iff for each $a\in A$ with $a\notin F$ and for all $\alpha\in K$, $\alpha\cup[F]$ is improper or $\alpha\nsubseteq[a]$ [21].*

Theorem 3.5. *(cf. [23])*
(1) If (A,K) is ST_2 (or ΔT_2) and F is (strongly) closed, then $(A/F,L)$ is ST_2 (or ΔT_2).
(2) All objects (A,K) in $\mathbf{CHY}$ are KT_2.
(3) (A,K) in $\mathbf{CHY}$ is LT_2 iff A is a point or the empty set.
(4) (A,K) in $\mathbf{CHY}$ is MT_2 iff A is a point or the empty set.
(5) (A,K) in $\mathbf{CHY}$ is NT_2 iff for each distinct pair x and y in A, $[x]\cap[y]\notin K$.

Remark 3.6. (cf. [23])
(1) If a Cauchy space (A,K) is $LT_2(MT_2)$ then it is KT_2. However, the converse is not true, in general. For example, let $A=\{x,y\}$ and $K=\{[x],\ [y],\ [\emptyset]\}$. Then (A,K) is KT_2 but it is not $LT_2(MT_2)$.
(2) If a Cauchy space (A,K) is NT_2 then it is KT_2. However, the converse is not true, in general. For example, let $A=\{x,y\}$ and $K=\{[x],\ [y],\ [\{x,\ y\}],\ [\emptyset]\}$. Then (A,K) is KT_2 but it is not NT_2.
(3) If a Cauchy space (A,K) is $LT_2(MT_2)$ then it is NT_2. However, the converse is not true, in general. For example, let $A=\{x,y\}$ and $K=\{[x],\ [y],\ [\emptyset]\}$. Then (A,K) is NT_2 but it is not $LT_2(MT_2)$.

Theorem 3.6. *Let (A, K) be a Cauchy space and $B \subset A$.*

(1) If (A, K) is $Pre\overline{T}_2$, then (B, K_1) is also $Pre\overline{T}_2$.
(2) If (A, K) is $PreT_2'$, then (B, K_1) is also $PreT_2'$.

(3) If (A, K) is $\overline{T}_2$, then (B, K_1) is also $\overline{T}_2$.
(4) If (A, K) is T_2', then (B, K_1) is also T_2'.

Proof. Let $f : B \hookrightarrow A$ be the inclusion map defined by $f(x) = x$ for $x \in B$ and K_1 be the initial lift of $f : B \hookrightarrow (A, K)$.

(1) Suppose that (A, K) is $Pre\overline{T}_2$ and $x \in B$. By Definition 2.2 and Theorem 3.1(3), (B, K_1) is also $Pre\overline{T}_2$.

(2) Let (A, K) is $PreT_2'$ and x, y be any two distinct points of B. Since $B \subset A$ and (A, K) is $PreT_2'$, by Theorem 3.1 (4), we have $[x] \cap [y] \in K$ and $f([x] \cap [y])$=$f([x]) \cap f([y])$= $[x] \cap [y] \in K$. Hence by Definition 2.2, $[x] \cap [y] \in K_1$ and by Theorem 3.1 (4), (B, K_1) is $PreT_2'$.

(3) Suppose that (A, K) is $\overline{T}_2$ and x, y be any two distinct points of B. Since $B \subset A$ and (A, K) is $\overline{T}_2$, by Theorem 3.1 (5), we have $[x] \cap [y] \notin K$ and $f([x] \cap [y])$=$f([x]) \cap f([y])$= $[x] \cap [y] \notin K$. Hence by Definition 2.2, $[x] \cap [y] \notin K_1$ and by Theorem 3.1 (5), (B, K_1) is $\overline{T}_2$.

The proof (4) is similar to the proof of (2) by using Theorem 3.1 (6). □

4. T_3-Objects

We now recall, ([2, 7, 12]), various generalizations of the usual T_3 separation axiom to arbitrary set based topological categories and characterize each of them for the topological categories **CHY**.

Definition 4.1. (cf. [2, 7, 12]) Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a non-empty subset of B.

1. X is $S\overline{T_3}$ iff X is T_1 and X/F is $Pre\overline{T_2}$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
2. X is ST_3' iff X is T_1 and X/F is $PreT_2'$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
3. X is $\overline{T_3}$ iff X is T_1 and X/F is $Pre\overline{T_2}$ for all closed $F \neq \emptyset$ in $U(X)$.
4. X is T_3' iff X is T_1 and X/F is $PreT_2'$ for all closed $F \neq \emptyset$ in $U(X)$.
5. X is KT_3 iff X is T_1 and X/F is $Pre\overline{T_2}$ if it is T_1, where $F \neq \emptyset$ in $U(X)$.
6. X is LT_3 iff X is T_1 and X/F is $PreT_2'$ if it is T_1, where $F \neq \emptyset$ in $U(X)$.
7. X is ST_3 iff X is T_1 and X/F is ST_2 if it is T_1, where $F \neq \emptyset$ in $U(X)$.
8. X is ΔT_3 iff X is T_1 and X/F is ΔT_2 if it is T_1, where $F \neq \emptyset$ in $U(X)$.

Remark 4.1. 1. For the category $\boldsymbol{Top}$ of topological spaces, all of the T_3's reduce to the usual T_3 separation axiom (cf. [2, 12**?**]).

2. If $\mathcal{U} : \mathcal{E} \to \boldsymbol{B}$, where $\boldsymbol{B}$ is a topos [15], then Parts (1), (2), and (5)-(8)of Definition 4.1 still make sense since each of these notions requires only finite products and finite colimits in their definitions. Furthermore, if $\boldsymbol{B}$ has infinite products and infinite wedge products, then Definition 4.1 (4), also, makes sense.

Theorem 4.1. *(cf. [23])*

(1) (A, K) in **CHY** *is $S\overline{T}_3$ iff for each distinct pair x and y in A, $[x] \cap [y] \notin K$.*
(2) (A, K) in **CHY** *is ST_3' iff A is a point or the empty set.*

(3) (A, K) in **CHY** *is $\overline{T}_3$ iff for each distinct pair x and y in A, $[x] \cap [y] \notin K$.*
(4) (A, K) in **CHY** *is T_3' iff A is a point or the empty set.*
(5) (A, K) in **CHY** *is KT_3 iff for each distinct pair x and y in A, $[x] \cap [y] \notin K$.*
(6) (A, K) in **CHY** *is LT_3 iff A is a point or the empty set.*
(7) (A, K) in **CHY** *is ST_3 iff for each pair of distinct points x and y in A and for any $\alpha,\ \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$.*
(8) (A, K) in **CHY** *is ΔT_3 iff for each pair of distinct points x and y in A and for any $\alpha,\ \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$.*

Theorem 4.2. *If (A, K) is KT_3, then $(A/F, L)$ is KT_3.*

Proof. Suppose (A, K) is KT_3. Let a and b be any distinct pair of points in A/F. By Theorem 4.1 (5), we only need to show that $[a] \cap [b] \notin L$, where L is the structure on A/F induced by q. Suppose that $a \neq *$ and $[a], [*] \in L$ implies $\exists\, [a], [y] \in K$ such that $[a] \supseteq q([a])$, $[*] \supseteq q([y])$, and $x = qx = a$, $qy = *$ for any $y \in F$. If $[a] \cap [*] \in L$, then $[a] \cap [y] \in K$, by definition of the quotient map and Remark 3.5. But $[a] \cap [y] \notin K$ since (A, K) is KT_3. Hence $[a] \cap [*] \notin L$. Similarly, if $a \neq b \neq *$ and $[a], [b] \in L$ implies $\exists\, [a], [b] \in K$ such that $[a] \supseteq q([a])$, $[b] \supseteq q([b])$, and $x = qx = a$, $qb = b$. If $[a] \cap [b] \in L$, then $[a] \cap [b] \in K$, by definition of the quotient map and Remark 3.5. But $[a] \cap [b] \notin K$ since (A, K) is KT_3. Hence $[a] \cap [b] \notin L$.

Consequently for each distinct points a and b in A/F, we have $[a] \cap [b] \notin L$. Hence by Theorem 4.1 (5), $(A/F, L)$ is KT_3. □

Theorem 4.3. *If (A, K) is ΔT_3, then $(A/F, L)$ is ΔT_3.*

Proof. It follows from Theorem 4.2. □

5. T_4-Objects

We now recall various generalizations of the usual T_4 separation axiom to arbitrary set based topological categories that are defined in [2, 7, 12], and characterize each of them for the topological categories **CHY**.

Definition 5.1. (cf. [2, 7, 12]) Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be a topological functor and X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a non-empty subset of B.

1. X is $S\overline{T_4}$ iff X is T_1 and X/F is $S\overline{T_3}$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
2. X is ST_4' iff X is T_1 and X/F is ST_3' for all strongly closed $F \neq \emptyset$ in $U(X)$.
3. X is $\overline{T_4}$ iff X is T_1 and X/F is $\overline{T_3}$ for all closed $F \neq \emptyset$ in $U(X)$.
4. X is T_4' iff X is T_1 and X/F is T_3' for all closed $F \neq \emptyset$ in $U(X)$.
5. X is ΔT_4 iff X is T_1 and X/F is ΔT_3 if it is T_1, where $F \neq \emptyset$ in $U(X)$.
6. X is KT_4 iff X is T_1 and X/F is KT_3 if it is T_1, where $F \neq \emptyset$ in $U(X)$.
7. X is LT_4 iff X is T_1 and X/F is LT_2 if it is T_1, where $F \neq \emptyset$ in $U(X)$.

Remark 5.1. 1. For the category $\boldsymbol{Top}$ of topological spaces, all of the T_4's reduce to the usual T_4 separation axiom ([2, 7, 12]).

2. If $\mathcal{U} : \mathcal{E} \to \boldsymbol{B}$, where $\boldsymbol{B}$ is a topos [15], then Definition 5.1 still makes sense since each of these notions requires only finite products and finite colimits in their definitions.

Theorem 5.1. *(cf. [23])*

(1) (A, K) in **CHY** *is $S\overline{T}_4$ iff for each distinct pair x and y in A, $[x] \cap [y] \notin K$.*
(2) (A, K) in **CHY** *is ST_4' iff A is a point or the empty set.*

(3) (A, K) in **CHY** *is $\overline{T}_4$ iff for each distinct pair x and y in A, $[x] \cap [y] \notin K$.*
(4) (A, K) in **CHY** *is T_4' iff A is a point or the empty set.*

Theorem 5.2. *(A, K) in* **CHY** *is ΔT_4 iff for each pair of distinct points x and y in A and for any $\alpha,\ \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$.*

Proof. It follows from Definition 5.1 (5), Theorem 3.1 (1) and Theorem 4.3. □

Theorem 5.3. *(A, K) in* **CHY** *is KT_4 iff for each distinct pair x and y in A, $[x] \cap [y] \notin K$.*

Proof. It follows from Definition 5.1 (6), Theorem 3.1 (1) and Theorem 4.2. □

Theorem 5.4. *(A, K) in* **CHY** *is LT_4 iff A is a point or the empty set.*

Proof. It follows from Definition 5.1 (7) and Theorem 3.5 (3). □

Remark 5.2. Let (A,K) be a Cauchy space. It follows from Theorem 3.5, Theorem 4.1, Theorem 5.1, Theorem 5.2 and Theorem 5.3 that (A,K) is $NT2$ iff (A,K) is $\overline{S}T_3$ iff (A,K) is $\overline{T}_3$ iff (A,K) is KT_3 iff (A,K) is $\overline{S}T_4$ iff (A,K) is $\overline{T}_4$ iff (A,K) is KT_4 iff (A,K) is ΔT_4 iff for each distinct pair x and y in A, $[x]\cap[y]\notin K$.

Remark 5.3. Let (A,K) be a Cauchy space. It follows from Theorem 3.5, Theorem 4.1, Theorem 5.1 and Theorem 5.4 that (A,K) is ST_3' iff (A,K) is T_3' iff (A,K) is LT_2 iff (A,K) is MT_2 iff (A,K) is LT_3 iff (A,K) is ST_4' iff (A,K) is T_4' iff (A,K) is LT_4 iff A is a point or the empty set.

6. Tychonoff objects

In this section, the characterization of Tychonoff objects in this category is given. Furthermore, we investigate the relationships between Tychonoff objects and ST_2, ΔT_2, ST_3, ΔT_3, generalized separation properties and separation properties at a point p in this category.

Definition 6.1. (cf. [7, 12]) Let $\mathcal{U}:\mathcal{E}\longrightarrow \boldsymbol{Set}$ be a topological functor and X an object in $\mathcal{E}$ with $\mathcal{U}(X)=B$.

1. X is $\Delta T_{3\frac{1}{2}}$ iff X is a subspace of ΔT_4.
2. X is $ST_{3\frac{1}{2}}$ iff X is a subspace of ST_4.
3. X is $T'_{3\frac{1}{2}}$ iff X is a subspace of T_4'.
4. X is $ST'_{3\frac{1}{2}}$ iff X is a subspace of ST_4'.
5. X is $C\Delta T_{3\frac{1}{2}}$ iff X is a subspace of a compact ΔT_2.
6. X is $CST_{3\frac{1}{2}}$ iff X is a subspace of a compact ST_2.
7. X is $LT_{3\frac{1}{2}}$ iff X is a subspace of a compact T_2'.
8. X is $S\Delta T_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact ΔT_2.
9. X is $SST_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact ST_2.
10. X is $SLT_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact T_2'.

Remark 6.1. For the category $\boldsymbol{Top}$ of topological spaces, all six of the properties defined in Definition 6.1 are equivalent and reduce to the usual $T_{3\frac{1}{2}}$= Tychonoff, i.e, completely regular T_1 spaces [27], Remark 5.2, and Remark 6.2.

Lemma 6.1. *(cf. [21]) All objects in CHY are (strongly) compact.*

Theorem 6.1. *Let (A,K) be a Cauchy space. Then the followings are equivalent:*
(1) (A,K) is $\Delta T_{3\frac{1}{2}}$,
(2) (A,K) is $ST_{3\frac{1}{2}}$,
(3) (A,K) is $C\Delta T_{3\frac{1}{2}}$,
(4) (A,K) is $CS\Delta T_{3\frac{1}{2}}$,
(5) (A,K) is $S\Delta T_{3\frac{1}{2}}$,
(6) (A,K) is $SST_{3\frac{1}{2}}$,
(7) for each distinct pair x and y in A, we have $[x]\cap[y]\notin K$.

Proof. It follows from Corollary 3.1, Theorem 5.1, Theorem 5.2, Definition 6.1 and Lemma 6.1. □

Example 6.1. Let $X=\{a,b\}$, $\delta=\{(X,X),(\{a\},\{a\}),(\{b\},\{b\}),(X,\{a\}),(\{a\},X),(X,\{b\}),(\{b\},X)\}$ and $\delta_1=\{(X,X),(\{a\},\{a\}),(\{b\},\{b\}),(X,\{a\}),(\{a\},X),(X,\{b\}),(\{b\},X),(\{a\},\{b\}),(\{b\},\{a\})\}$. Then (X,δ) is $C\Delta T_{3\frac{1}{2}}$, but (X,δ_1) is not $C\Delta T_{3\frac{1}{2}}$, since $(\{a\},\{b\})\in\delta$ with $a\neq b$.

Theorem 6.2. *Let (A, K) be a Cauchy space. Then the followings are equivalent:*

(1) (A, K) is $T'_{3\frac{1}{2}}$,

(2) (A, K) is $ST'_{3\frac{1}{2}}$,

(3) A is a point or the empty set.

Proof. It follows from Theorem 5.1 and Definition 6.1. □

Theorem 6.3. *Let (A, K) be a Cauchy space. Then the followings are equivalent:*

(1) (A, K) is $LT_{3\frac{1}{2}}$,

(2) (A, K) is $SLT_{3\frac{1}{2}}$,

(3) for each pair of distinct points x and y in A, we have $[x] \cap [y] \in K$(equivalently, for each finite subset F of A, we have $[F] \in K$).

Proof. It follows from Theorem 3.1, Definition 6.1 and Lemma 6.1. □

We can infer the following results.

Remark 6.2. Let (A, K) be in **CHY**. The followings are equivalent;

1. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, (A, K) is T_1 iff it is T_0 iff it is $\overline{T}_0$ iff (A, K) is $S\overline{T}_3$ iff it is $\overline{T}_3$ iff it is KT_3 iff (A, K) is $S\overline{T}_4$ iff it is $\overline{T}_4$ iff (A, K) is ST_2 or ΔT_2 iff (A, K) is ST_3 or ΔT_3 iff (A, K) is $NT2$ iff (A, K) is $\Delta T_{3\frac{1}{2}}$ iff (A, K) is $ST_{3\frac{1}{2}}$ iff (A, K) is $C\Delta T_{3\frac{1}{2}}$ iff (A, K) is $CS\Delta T_{3\frac{1}{2}}$ iff (A, K) is $S\Delta T_{3\frac{1}{2}}$ iff (A, K) is $SST_{3\frac{1}{2}}$.

2. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, (A, K) is $\overline{T}_2$ iff (A, K) is $S\overline{T}_3$ iff (A, K) is $\overline{T}_3$ iff (A, K) is KT_3 iff (A, K) is $S\overline{T}_4$ iff (A, K) is $\overline{T}_4$ iff (A, K) is ST_2 or ΔT_2 iff (A, K) is ST_3 or ΔT_3 iff (A, K) is $NT2$ iff (A, K) is $\Delta T_{3\frac{1}{2}}$ iff (A, K) is $ST_{3\frac{1}{2}}$ iff (A, K) is $C\Delta T_{3\frac{1}{2}}$ iff (A, K) is $CS\Delta T_{3\frac{1}{2}}$ iff (A, K) is $S\Delta T_{3\frac{1}{2}}$ iff (A, K) is $SST_{3\frac{1}{2}}$.

3. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, if (A, K) is $S\overline{T}_3$ or $\overline{T}_3$ or KT_3 or $S\overline{T}_4$ or $\overline{T}_4$ or ST_2 or ΔT_2 or ST_3 or ΔT_3 or $NT2$ or $\Delta T_{3\frac{1}{2}}$ or $ST_{3\frac{1}{2}}$ or $C\Delta T_{3\frac{1}{2}}$ or $CS\Delta T_{3\frac{1}{2}}$ or $S\Delta T_{3\frac{1}{2}}$ or $SST_{3\frac{1}{2}}$, then (A, K) is T'_0. But the converse of implication is not true, in general. For example, let $A = \{x, y\}$ and $K = \{[x], [y], [\{x, y\}], [\emptyset]\}$. Then (A, K) is T'_0 but it is not $S\overline{T}_3$ or $\overline{T}_3$ or KT_3 or $S\overline{T}_4$ or $\overline{T}_4$ or ST_2 or ΔT_2 or ST_3 or ΔT_3 or $NT2$ or $\Delta T_{3\frac{1}{2}}$ or $ST_{3\frac{1}{2}}$ or $C\Delta T_{3\frac{1}{2}}$ or $CS\Delta T_{3\frac{1}{2}}$ or $S\Delta T_{3\frac{1}{2}}$ or $SST_{3\frac{1}{2}}$.

4. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, if (A, K) is $S\overline{T}_3$ or $\overline{T}_3$ or KT_3 or $S\overline{T}_4$ or $\overline{T}_4$ or ST_2 or ΔT_2 or ST_3 or ΔT_3 or $NT2$ or $\Delta T_{3\frac{1}{2}}$ or $ST_{3\frac{1}{2}}$ or $C\Delta T_{3\frac{1}{2}}$ or $CS\Delta T_{3\frac{1}{2}}$ or $S\Delta T_{3\frac{1}{2}}$ or $SST_{3\frac{1}{2}}$, then (A, K) is $Pre\overline{T}_2$. But the converse of implication is not true, in general. For example, let $A = \{x, y\}$ and $K = \{[x], [y], [\{x, y\}], [\emptyset]\}$. Then (A, K) is $Pre\overline{T}_2$ but it is not $S\overline{T}_3$ or $\overline{T}_3$ or KT_3 or $S\overline{T}_4$ or $\overline{T}_4$ or ST_2 or ΔT_2 or ST_3 or ΔT_3 or $NT2$ or $\Delta T_{3\frac{1}{2}}$ or $ST_{3\frac{1}{2}}$ or $C\Delta T_{3\frac{1}{2}}$ or $CS\Delta T_{3\frac{1}{2}}$ or $S\Delta T_{3\frac{1}{2}}$ or $SST_{3\frac{1}{2}}$.

5. By Theorems 3.1, 4.1 and 6.2, Corollary 3.1, the followings are equivalent:

(a) (A, K) is $PreT'_2$ (T'_2), $LT_{3\frac{1}{2}}$, $SLT_{3\frac{1}{2}}$, and is $S\overline{T}_3$ or $\overline{T}_3$ or KT_3 or $S\overline{T}_4$ or $\overline{T}_4$ or ST_2 or ΔT_2 or ST_3 or ΔT_3 or $NT2$ or $\Delta T_{3\frac{1}{2}}$ or $ST_{3\frac{1}{2}}$ or $C\Delta T_{3\frac{1}{2}}$ or $CS\Delta T_{3\frac{1}{2}}$ or $S\Delta T_{3\frac{1}{2}}$ or $SST_{3\frac{1}{2}}$.

(b) A is a point or the empty set.

6. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1 and Definition 3.2, (A, K) is $S\overline{T}_3$ or $\overline{T}_3$ or KT_3 or $S\overline{T}_4$ or $\overline{T}_4$ or ST_2 or ΔT_2 or ST_3 or ΔT_3 or $NT2$ or $\Delta T_{3\frac{1}{2}}$ or $ST_{3\frac{1}{2}}$ or $C\Delta T_{3\frac{1}{2}}$ or $CS\Delta T_{3\frac{1}{2}}$ or $S\Delta T_{3\frac{1}{2}}$ or $SST_{3\frac{1}{2}}$ iff (A, K) is $\mathbf{T}_2$ (we will refer to it as the usual one).

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